Weak-coupling expansion for the pressure of hot QCD — Introduction —

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1. What is it?

2. Why is it relevant?

3. Where are we now?

4. What should be done?

QCD:

$$\mathcal{L}_{\rm QCD} = \frac{1}{4g^2}\sum_{a=1}^{N_{\rm c}^2-1} F^a_{\mu\nu}F^a_{\mu\nu} + \sum_{i=1}^{N_{\rm f}} \bar\psi_i[\gamma_\mu D_\mu + m_i]\psi_i \; .$$

Thermodynamics:

Minus grand canonical free energy density, i.e. pressure.

$$p(T,\boldsymbol{\mu}) \equiv \lim_{V\to\infty}\frac{T}{V}\ln\left\{ {\rm Tr}\left[\exp\left(-\frac{\mathcal{H}_{\rm QCD}-\mu_i\mathcal{Q}_i}{T}\right)\right]\right\} \; ,$$

where $\mathcal{H}_{\rm QCD}$ is the Hamilton operator, and $\mathcal{Q}_i$ are quark number operators. Let $p(T)\equiv p(T,\mathbf{0})$.

Why is it relevant?

In cosmology, the cooling rate of the Universe is

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} = -\frac{\sqrt{24\pi}}{m_{\mathrm{Pl}}} \frac{\sqrt{e(T)}}{\mathrm{d}\ln s(T)/\mathrm{d}T} ,$$

where $s(T) = \mathrm{d}p(T)/\mathrm{d}T$, $e(T) = Ts(T) - p(T)$.

Cosmological relics (dark matter, background radiation, etc) are born when some reaction time $\tau(T)$ becomes longer than the age of the Universe $t(T)$.

For instance, WIMPs of mass m decouple at $T \sim m/25$. For $m =$ 10...1000 GeV, $T =$ 0.4...40 GeV, in which range QCD dominates the equation of state.

Hindmarsh, Philipsen 2005

Or: right-handed "sterile" neutrinos with $m_\nu \sim$ keV can be warm dark matter decoupling at $T \sim 150$ MeV.

Abazajian, Fuller 2002

Asaka, Shaposhnikov 2005

The dark matter relic density is determined (indirectly) within few % by forthcoming CMB experiments, so theoretical errors should be reduced to the same level.

In heavy ion collision experiments, the expansion of the system, after thermalisation, is determined by

$$T^{\mu\nu} = [p(T) + e(T)]u^\mu u^\nu - p(T)g^{\mu\nu}, \quad \partial_\mu T^{\mu\nu} = 0 \,,$$

where $u^\mu(t, \mathbf{x})$ is the flow velocity, and $T = T(t, \mathbf{x})$.

After hydrodynamic expansion the system hadronises at $T \sim 100...150$ MeV. The hadron spectrum observed depends indirectly on $p(T)$.

In particular, certain observables like the “elliptic flow” are supposed to probe early stages with $T \gtrsim 200$ MeV.

Current status

$T \gg 200$ MeV $\Rightarrow$ asymptotic freedom $\Rightarrow$ weakly interacting partons, described by QCD pert. theory.

$$p_{\mathsf{SB}}(T) \equiv \frac{\pi^2 T^4}{90}\left[2(N_c^2-1)+\frac{7}{2}N_c N_f\right] \approx 5.2T^4 \,.$$

But one can also compute corrections to $p_{\mathsf{SB}}(T)$ in a power series in the QCD coupling constant g.

g^2:	Shuryak 1978; Chin 1978
g^3:	Kapusta 1979
$g^4 \ln(1/g)$:	Toimela 1983
g^4:	Arnold, Zhai 1994
g^5:	Zhai, Kastening 1995; Braaten, Nieto 1995
$g^6 \ln(1/g)$:	Schröder 2002; Kajantie et al 2002

$\therefore$ The structure of the perturbative series is non-trivial.

The reason: interactions make it a **multiscale system**, generating colour-electric screening at $|\mathbf{k}| \sim m_\mathrm{E} \sim gT$, and colour-magnetic screening at $|\mathbf{k}| \sim g^2 T$.

Expansion parameter:

$$\epsilon \sim g^2 n_b(|\mathbf{k}|) = \frac{g^2}{\exp(|\mathbf{k}|/T) - 1} \overset{|\mathbf{k}| \lesssim T}{\approx} \frac{g^2 T}{|\mathbf{k}|} \, .$$

Method for treating this: **effective field theories**.

QCD; $|\mathbf{k}| \sim 2\pi T, gT, g^2T$

$\Downarrow$ perturbation theory (1)

EQCD; $|\mathbf{k}| \sim gT, g^2T$

$\Downarrow$ perturbation theory (2)

MQCD; $|\mathbf{k}| \sim g^2T$

$\Downarrow$ numerical simulations (3)

$p(T)$.

Contributions: ($\Lambda_\mathrm{E}, \Lambda_\mathrm{M}$ are "matching scales")

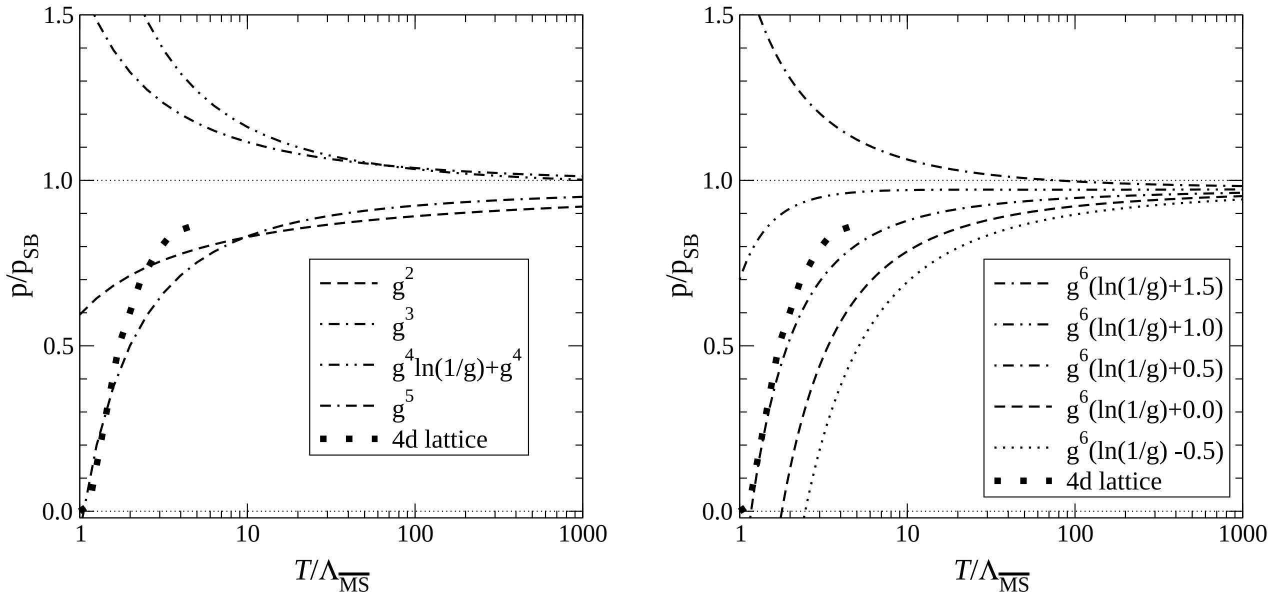
$$\frac{\delta p_{(1)}}{T^4} \sim 1 + g^2 + g^4 \ln\frac{2\pi T}{\Lambda_\mathrm{E}} + g^6 \ln\frac{2\pi T}{\Lambda_\mathrm{E}} + \ldots\,,$$

$$\frac{\delta p_{(2)}}{T^4} \sim g^3 + g^4 \ln\frac{\Lambda_\mathrm{E}}{m_\mathrm{E}} + g^5 + g^6 \ln\frac{\Lambda_\mathrm{E}}{m_\mathrm{E}} + g^6 \ln\frac{m_\mathrm{E}}{\Lambda_\mathrm{M}} + \ldots$$

$$\frac{\delta p_{(3)}}{T^4} \sim g^6 \left(\ln\frac{\Lambda_\mathrm{M}}{g^2 T} + [\text{non-pert}] \right)\,.$$

Status: coefficients up to 4-loop logarithms are known analytically, but some "constant" 4-loop terms not.

Numerical evaluation ($N_\mathrm{f} = 0$):



Kajantie, Laine, Rummukainen, Schröder 2002

$\Rightarrow$ Interactions are strong even at high temperatures (sQCD): full order g^6 might work, but need not to.

So let us try a more conservative approach.

Only carry out step (1) analytically, and study

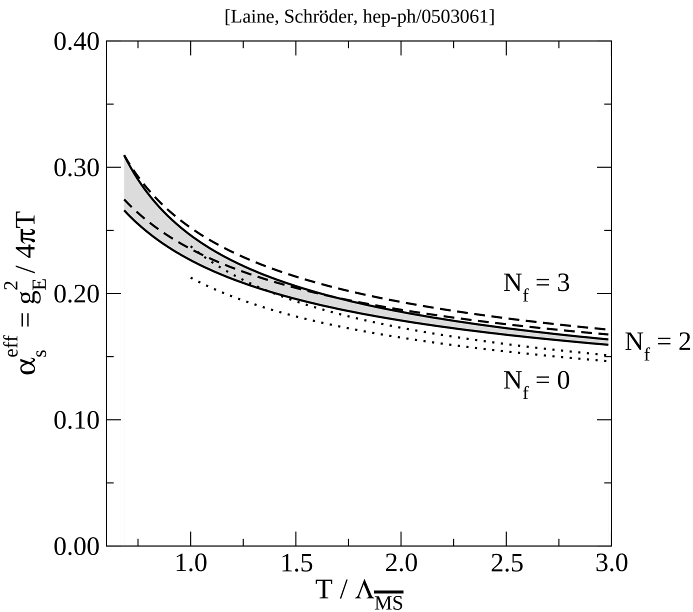
$$S_{\mathsf{EQCD}} = \int \mathrm{d}^3\mathbf{x}\,\mathcal{L}_{\mathsf{EQCD}}\ ,$$

$$\mathcal{L}_{\mathsf{EQCD}} = \tfrac{1}{2}\,\mathrm{Tr}[F_{ij}^2] + \mathrm{Tr}[D_i, A_0]^2 + m_{\mathsf{E}}^2\,\mathrm{Tr}[A_0^2] + \lambda_{\mathsf{E}}(\mathrm{Tr}[A_0^2])^2\ ,$$

where $F_{ij} = (i/g_{\mathsf{E}})[D_i, D_j]$, $D_i = \partial_i - i g_{\mathsf{E}} A_i$, $A_i = A_i^a T^a$, and $A_0 = A_0^a T^a$, non-perturbatively.

[In practice "non-perturbatively" probably means "numerically", although it does not need to be so].

This should be a safe procedure, down to low T:



On the contrary, perturbation theory with EQCD, leading to odd powers, need not converge in general.

The master formula for the physical pressure:

$$\frac{p_{\mathsf{QCD}}}{T^4} = 1+g^2+g^4\Big(\ln\tfrac{2\pi T}{\Lambda_{\mathsf{E}}}+C_4\Big)+g^6\Big(\ln\tfrac{2\pi T}{\Lambda_{\mathsf{E}}}+C_6\Big)+\mathcal{O}(g^8)+$$

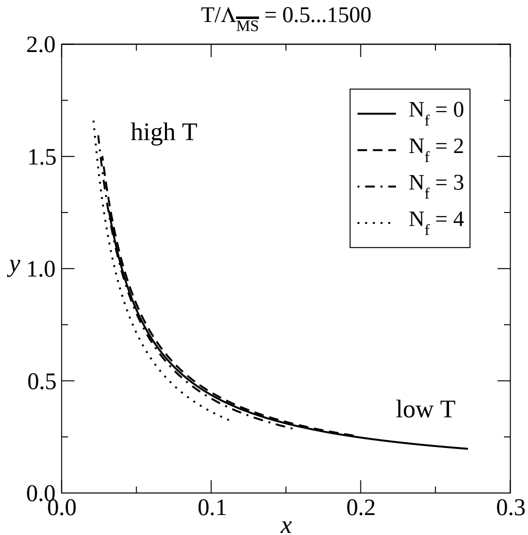
$$+\ \frac{p_{\mathsf{EQCD}}}{T^4}\,,$$

$$\frac{p_{\mathsf{EQCD}}}{T^4} = \lim_{V\to\infty}\tfrac{1}{VT^3}\ln\Big[\int_{\Lambda_{\mathsf{E}}}\mathcal{D}A_i\mathcal{D}A_0\exp\big(-S_{\mathsf{EQCD}}\big)\Big]\,.$$

Here p_{EQCD}/T^4 depends “trivially” on $\Lambda_{\mathsf{E}}, g_{\mathsf{E}}^2$, and non-trivially on the dimensionless combinations

$$x \equiv \frac{\lambda_{\mathsf{E}}}{g_{\mathsf{E}}^2}\,,\qquad y \equiv \frac{m_{\mathsf{E}}^2}{g_{\mathsf{E}}^4}\,.$$

We need to know p_{EQCD}/T^4 in a part of the (x, y)-plane.



Absolute value not measurable, but derivatives are:

$$\frac{\partial}{\partial y}\left\{\frac{p_{\mathsf{EQCD}}}{T^4}\right\} \propto \left\langle \frac{\mathrm{Tr}[A_0^2]}{g_{\mathsf{E}}^2} \right\rangle , \quad \frac{\partial}{\partial x}\left\{\frac{p_{\mathsf{EQCD}}}{T^4}\right\} \propto \ldots .$$

Challenge 1: determine C_6.

- Compute 4-loop p_{QCD} in full QCD with a certain UV and IR regularization.

- Subtract 4-loop p_{EQCD} computed with the same UV and IR regularization.

- If we use **dimensional regularization** as the only UV and IR regularization, then p_{EQCD} vanishes exactly, and we "only" need to do un-resummed 4-loop sum-integrals on the full QCD side.

$\Rightarrow$ talks by Y. Schröder, A. Vuorinen

Challenge 2: To obtain p_{EQCD} from $\partial_y\{p_{\text{EQCD}}\}$, $\partial_x\{p_{\text{EQCD}}\}$, we need to fix the integration constant.

- This can be achieved by going to $y \to \infty$, where steps (2), (3) can be reliably carried out after all: we are then left with $\mathcal{L}_{\text{MQCD}} = \frac{1}{2}\,\mathrm{Tr}[F_{ij}^2] + \ldots$.

  $\Rightarrow$ talk by P. Giovannangeli

- The resulting contribution has been determined non-perturbatively on the lattice, but a conversion to dimensional regularization remains to be completed.

  $\Rightarrow$ talks by F. Di Renzo, C. Torrero

- Suggestion by M. Shaposhnikov: could one not stay within EQCD, and consider $y \to -\infty$, for $N_c = 2$?

Challenge 3: In order to use this philosophy, we need to measure $\partial_y\{p_{\mathrm{EQCD}}\}$, $\partial_x\{p_{\mathrm{EQCD}}\}$ up to $y \gg 1$.

- However there is then a heavy scalar on the lattice.

- We need a small lattice spacing to account for it correctly: $a \ll 2\pi/M$.

$\Rightarrow$ talk by A. Hietanen

- But how small is small enough? Need to know whether there can be $\mathcal{O}(a \ln a)$ effects, and it would also help to compute $\mathcal{O}(a)$ effects explicitely.

$\Rightarrow$ for possible techniques, guest talk by A. Rago

Further challenges

If all of this could be achieved, and maybe even before that, one might also consider (in arbitrary order):

Dependence on N_c:
Would something be simpler for $N_\mathrm{c} = 2$?
Would something be simpler for $N_\mathrm{c} \gg 3$?

Dependence on N_f:
We may expect better lattice data at $N_\mathrm{f} > 0$ in the near future, hopefully with Wilson but certainly with "Asqtad"-improved staggered quarks.

"Though purists howled — and are still howling — the technique works" [New Scientist, 13 Aug 2005].

“Trivially”: behaviour of observables other than $p(T)$.

- Entropy density, $s(T) = \mathrm{d}p(T)/\mathrm{d}T$.

⇒ Blaizot, Iancu, Rebhan

- For comparison with lattice and possibly for applications in cosmology: interaction measure / trace anomaly,

$$T^5 \frac{\mathrm{d}}{\mathrm{d}T}\left[\frac{p(T)}{T^4}\right] = e(T) - 3p(T) \propto \beta_0 T^4 \; .$$

Murayama et al, hep-ph/0403019.

Dependence on chemical potentials

- Susceptibilities $\frac{\partial^2}{\partial\mu_i\partial\mu_j}p(T,\boldsymbol{\mu})$.

To $g^6\ln(1/g)$: Vuorinen 2003

- The complete function $p(T,\boldsymbol{\mu})$.

Ipp et al, in preparation

At least susceptibilities should in the end be extended to the full order g^6.

Dependence on quark masses

- Susceptibilities $\frac{\partial^2}{\partial m_i \partial m_j} p(T)$.

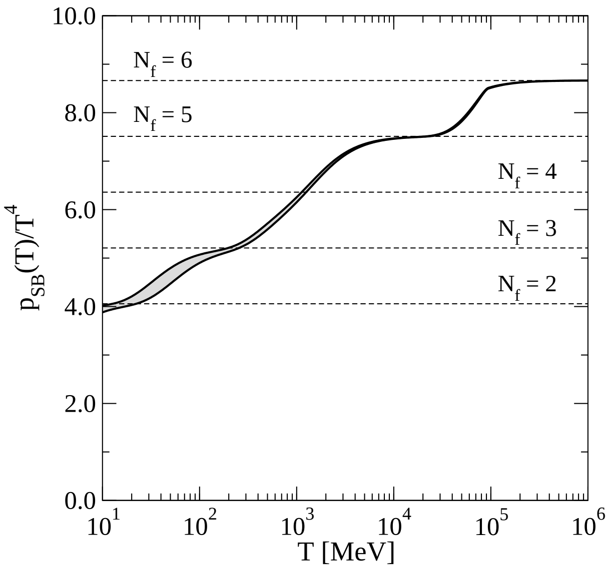
To $g^6 \ln(1/g)$: Laine and Schröder, in preparation

- The whole dependence of $p(T)$ on quark masses would be as important as $p(T, \mu)$, but may technically be even harder (at least as far as analytic results are concerned).

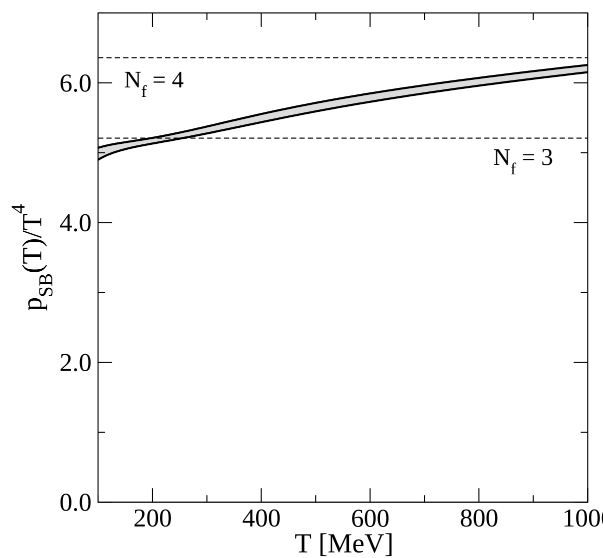
Is even $\mathcal{O}(g^2)$ in the literature?

Indeed, one naively thinks it's enough to interpolate between various N_f's, but that's not really the case.

SB for Cosmology:



SB for Heavy Ions:



Conclusions

There is still work to do.

Organizational

Coffee breaks: 10.30, 15.30.

Lunches: 12.30.

Dinners: 19.30.

Fee: 60€.