NUMERICAL STOCHASTIC P.T. for FULL QCD

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Plan of the talk

- MOTIVATIONS (LPT and its difficulties)
- STOCHASTIC QUANTIZATION → SPT
- NSPT (Numerical Stoch. Pert. T.) for LGT
- Adding FERMIONS, i.e. NSPT for FULL QCD
- APPLICATIONS : δm in LHQET ($\to m_b$)
  ($O(\alpha^3)$ computations)
  CRITICAL MASS for Wilson fermion
  Z_q in RI' (as a prototype)
- Conclusions and perspectives

LATTICE PERTURBATION THEORY and its DIFFICULTIES

In principle, the LATTICE is a REGULATOR like any other in PERT. THEORY... In practice...

(a) Lattice regularization of field theory has been mainly motivated by non perturbative issues: FORMULATION itself of FT; feasibility of COMPUTER SIMULATIONS.

(b) Still, LPT is something you have to live with anyway... Typical (historical...) working grounds:

- MATCHING (renormalization const's Z's ; effective theories)
- SYMANZIK IMPROVEMENT program coefficients

$$S_{eff} = \int dx \left(\mathcal{L}_0(x) + a\,\mathcal{L}_1(x) + a^2 \mathcal{L}_2(x) + \dots \right) \quad \text{(lattice as an EFF.T.)}$$

$$\phi_{eff} = \phi_0 + a\,\phi_1 + a^2 \phi_2 + \dots$$

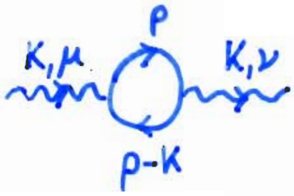
GOAL: compute IMPROV. COEFF's so that OBSERVABLES are affected by $O(a^2)$ (not $O(a)$) $\rightarrow$ FASTER CONT. LIMIT...

LQCD : $S_W + c_{SW}\, S_{clover}$ and (i.e.) $A^a_{I\mu} = A^a_\mu + c_A \hat{\partial}_\mu P^a$

(b') Many progresses have in recent years been made possible by the application of completely NON PERTURBATIVE methods!

Still
- severe limitations are now posed by the UNQUENCHING program for LGT!
- the perturbative regime is of course a firm REFERENCE POINT...

(c) As a matter of fact, LPT is HARD ... actually harder than any other perturbative regularization...



$$= -\int_{BZ} \frac{d^4p}{(2\pi)^4}\, \mathrm{Tr}\left(V_\nu(2p-K)\, S(p)\, V_\mu(2p-K)\, S(p-K)\right)$$

$$S(p) = \frac{-i\sum_\mu \gamma_\mu \sin p_\mu + M(p)}{\sum_\mu \sin^2 p_\mu + M^2(p)} \qquad M(p) = M + 2r\sum_\mu \sin^2\frac{p_\mu}{2}$$

$$V_\mu = -ie\,(2\pi)^4\,\delta_P(p-p'+K)\left[\gamma_{\mu\,\alpha\beta}\cos\frac{p_\mu+p'_\mu}{2} - ir\,\delta_{\alpha\beta}\sin\frac{p_\mu+p'_\mu}{2}\right]$$

- continuum standard tricks not so useful...
- coordinate space methods (Lüscher & Weisz)
- lots of COMPUTER ALGEBRA

  and HIGHLY NON TRIVIAL NUMERICS ...

It is worth having a closer look at Wilson action for LGT...

↳

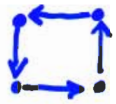
- **Perturbation Theory for LATTICE GAUGE THEORY**

Wilson ACTION

Wilson '74

$$\boxed{S_G = -\frac{\beta}{2N}\sum_p \mathrm{tr}\,(U_p + U_p^\dagger)}$$

gauge invariant at every value of CUTOFF!

formulated in terms of $U_\mu(x) = e^{gA_\mu(x)}$ $\quad x \xrightarrow{U_\mu(x)} x+\mu \qquad U_p \sim$ 

and you want to evaluate weak coupling Pert. Expansions of quantities like

$$\boxed{\langle O \rangle = Z^{-1}\int DU\, e^{-S_G[U]}\, O[U]}$$

Now, Pert. Th. is given in terms of A_μ ... (GROUP → ALGEBRA)

THE PROGRAM

① $DU \rightsquigarrow DA\; e^{-S_m[A]}$ ("interaction from the measure"...)

② FADDEEV & POPOV $\quad Z^{-1}\int DA\, D\bar{c}\, Dc\; e^{-S_G[A]-S_m[A]-S_{FP}[A,\bar{c},c]-S_{GF}[A]}\, O[A]$

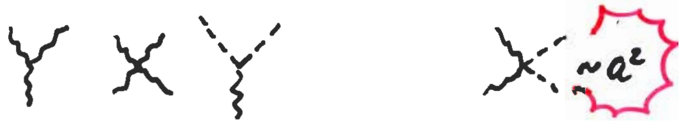
(actually $Z^{-1}\int DA\; e^{-S_G[A]-S_m[A]-S_{GF}[A]}\, O[A]\, \det FP(A)$)

③ PROPAGATORS and VERTICES

↳ NOT GIVEN ONCE AND FOR ALL

↳ BOTH RELEVANT and IRRELEVANT

EX. PURE GAUGE at 1 LOOP



$\sim a^2$

$\sim \frac{1}{a^2}$

From a textbook (Rothe) : LIFE in LPT is HARD

Hence in the continuum limit, (14.43) reduces to the familiar expression of the continuum formulation

$$\Gamma^{ABC}_{\mu\nu\lambda}(k,k',k'')\underset{a\to 0}{\longrightarrow} ig_0(2\pi)^4\delta^{(4)}(k+k'+k'')f_{ABC}\left[(k''-k')_\mu\delta_{\nu\lambda}\right.$$
$$\left.+(k-k'')_\nu\delta_{\mu\lambda}+(k'-k)_\lambda\delta_{\mu\nu}\right].$$

Needless to say, the calculation of the four gluon vertex (see fig. on page 212) from the fourth order contribution in θ_i^A to the effective action is quite tedious and we shall not present it here. The expression is very lengthy and has been given in the appendix of the paper by Kawai et al. (1981):*

— and you can even

$$\Gamma^{ABCD}_{\mu\nu\lambda\rho}(p,q,r,s)=$$
$$-g^2 f_{ABE}f_{CDE}\Big\{\delta_{\mu\lambda}\delta_{\nu\rho}\left[\cos\frac{a(q-s)_\mu}{2}\cos\frac{a(p-r)_\nu}{2}-\frac{a^4}{12}\hat{p}_\nu\hat{q}_\mu\hat{r}_\nu\hat{s}_\mu\right]$$
$$-\delta_{\mu\rho}\delta_{\nu\lambda}\left[\cos\frac{a(q-r)_\mu}{2}\cos\frac{a(p-s)_\nu}{2}-\frac{a^4}{12}\hat{p}_\nu\hat{q}_\mu\hat{r}_\mu\hat{s}_\nu\right]$$
$$+\frac{a^2}{12}\delta_{\mu\nu}\delta_{\mu\lambda}\delta_{\mu\rho}\sum_\sigma(\hat{q}_\sigma e^{-i\frac{a}{2}p_\sigma}-\hat{p}_\sigma e^{-i\frac{a}{2}q_\sigma})(\hat{s}_\sigma e^{-i\frac{a}{2}r_\sigma}-\hat{r}_\sigma e^{-i\frac{a}{2}s_\sigma})$$
$$-\frac{a^2}{6}\delta_{\mu\nu}\delta_{\mu\lambda}(\hat{q}_\rho e^{-i\frac{a}{2}p_\rho}-\hat{p}_\rho e^{-i\frac{a}{2}q_\rho})\hat{s}_\mu\cos\frac{ar_\rho}{2}$$
$$+\frac{a^2}{6}\delta_{\mu\nu}\delta_{\mu\rho}(\hat{q}_\lambda e^{-i\frac{a}{2}p_\lambda}-\hat{p}_\lambda e^{-i\frac{a}{2}q_\lambda})\hat{r}_\mu\cos\frac{as_\lambda}{2}$$
$$-\frac{a^2}{6}\delta_{\mu\lambda}\delta_{\mu\rho}(\hat{s}_\nu e^{-i\frac{a}{2}r_\nu}-\hat{r}_\nu e^{-i\frac{a}{2}s_\nu})\hat{q}_\mu\cos\frac{ap_\nu}{2}$$
$$+\frac{a^2}{6}\delta_{\nu\lambda}\delta_{\nu\rho}(\hat{s}_\mu e^{-i\frac{a}{2}r_\mu}-\hat{r}_\mu e^{-i\frac{a}{2}s_\mu})\hat{p}_\nu\cos\frac{aq_\mu}{2}\Big\}$$
$$+(B\leftrightarrow C,\nu\leftrightarrow\lambda,q\leftrightarrow r)+(B\leftrightarrow D,\nu\leftrightarrow\rho,q\leftrightarrow s)$$
$$+g^2\frac{a^4}{12}\Big\{\frac{2}{3}(\delta_{AB}\delta_{CD}+\delta_{AC}\delta_{BD}+\delta_{AD}\delta_{BC})$$
$$+(d_{ABE}d_{CDE}+d_{ACE}d_{BDE}+d_{ADE}d_{BCE})\Big\}\Big\{\delta_{\mu\nu}\delta_{\mu\lambda}\delta_{\mu\rho}\sum_\sigma\hat{p}_\sigma\,\hat{q}_\sigma\hat{r}_\sigma\hat{s}_\sigma$$
$$-\delta_{\mu\nu}\delta_{\mu\lambda}\hat{p}_\rho\hat{q}_\rho\hat{r}_\rho\hat{s}_\mu-\delta_{\mu\nu}\delta_{\mu\rho}\hat{p}_\lambda\hat{q}_\lambda\hat{s}_\lambda\hat{r}_\mu$$
$$-\delta_{\mu\lambda}\delta_{\mu\rho}\hat{p}_\nu\hat{r}_\nu\hat{s}_\nu\hat{q}_\mu-\delta_{\nu\lambda}\delta_{\nu\rho}\hat{q}_\mu\hat{r}_\mu\hat{s}_\mu\hat{p}_\nu$$
$$+\delta_{\mu\nu}\delta_{\lambda\rho}\hat{p}_\lambda\hat{q}_\lambda\hat{r}_\mu\hat{s}_\mu+\delta_{\mu\lambda}\delta_{\nu\rho}\hat{p}_\nu\hat{r}_\nu\hat{q}_\mu\hat{s}_\mu+\delta_{\mu\rho}\delta_{\nu\lambda}\hat{p}_\nu\hat{s}_\nu\hat{q}_\mu\hat{r}_\mu\Big\}\qquad(14.44)$$

GO WRONG...

* The expression given in the above reference is however not completely correct. We give here the corrected form which was provided to us by W. Wetzel.

- Numerical Stochastic Perturbation Theory : a brief introduction
  (Di Renzo, Marchesini, Onofri '93)

NSPT comes (almost for free...) as an application of STOCHASTIC QUANTIZATION...
(Parisi, Wu '81)

0. QUANTUM THEORY $\langle O[\phi]\rangle = \dfrac{\int D\phi\, e^{-S[\phi]}\, O[\phi]}{\int D\phi\, e^{-S[\phi]}}$

1. $\phi(x) \longrightarrow \phi_\eta(x,\tau)$ with $\dfrac{d}{d\tau}\phi_\eta(x,\tau) = -\dfrac{\delta S[\phi]}{\delta\phi} + \eta(x,\tau)$ ◊

(LANGEVIN equation) GAUSSIAN NOISE $\langle \eta_i \eta_j \rangle_\eta = 2\,\delta_{ij}$

2. ASYMPTOTICALLY $\langle O[\phi_\eta(x,\tau)]\rangle_\eta \xrightarrow[\tau\to\infty]{} \langle O[\phi(x)]\rangle$

NSPT : Since $\dfrac{\delta S}{\delta \phi}$ depends on the coupling g $\phi_\eta = \phi_\eta(g)$

THEN EXPAND $\phi_\eta = \phi_\eta^{(0)} + g\,\phi_\eta^{(1)} + g^2\phi_\eta^{(2)} + \ldots + g^k\phi_\eta^{(k)} + \ldots$

L.eq. ◊ gets translated in HIERARCHY of EQUATIONS exactly TRUNCABLE

NOW $\phi \rightarrow \{\phi_\eta^{(i)}\}$ $O[\phi] \longrightarrow O[\{\phi_\eta^{(i)}\}] \sim \sum_k \{O_\eta^{(k)}\}$

and (for example...)

$$\langle \phi(x)\,\phi(y)\rangle^{(\mathrm{ord}\ 2)} = \lim_{\tau\to\infty} \langle \phi_\eta^{(0)}(x,\tau)\phi_\eta^{(2)}(y,\tau) + \phi_\eta^{(1)}(x,\tau)\phi_\eta^{(1)}(y,\tau) + \phi_\eta^{(2)}(x,\tau)\phi_\eta^{(0)}(y,\tau)\rangle$$

REMARK $\langle O[\phi_\eta(t)]\rangle_\eta \equiv \int D\phi \, P[\phi,t] O[\phi]$ ... $P[\phi,t] \xrightarrow[t\to\infty]{} Z^{-1} e^{-S[\phi]}$

STOCHASTIC PERTURBATION THEORY

EX $S[\phi] = \int dx \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right)$ $\dot\phi(x,t) = (\partial^2 - m^2)\phi - \frac{\lambda}{3!}\phi^3 + \eta$

- **FREE FIELD** $\dot\phi(k,t) = -(k^2+m^2)\phi(k,t) + \eta(k,t)$

you can find the solution

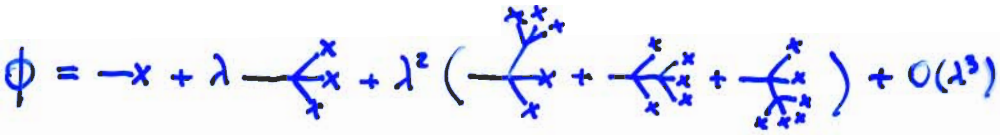
$$\phi(k,t) = \int_0^t d\tau \, G(k,t-\tau)\eta(k,\tau) = \int_0^t d\tau \, e^{-(k^2+m^2)(t-\tau)} \eta(k,\tau)$$

- **INTER. FIELD** $\eta(k,t) \Rightarrow \eta(k,t) - \frac{\lambda}{3!}\int \frac{dq\,dp}{(2\pi)^{2n}} \phi(p,t)\phi(q,t)\phi(k-p-q)$

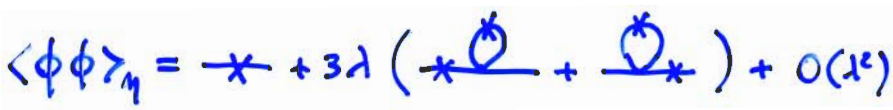
which means $\phi(k,t) = \int_0^t d\tau \, e^{-(k^2+m^2)(t-\tau)} \left[\eta(k,\tau) - \frac{\lambda}{3!}\int \frac{dp\,dq}{(2\pi)^{2n}} \phi(p,\tau)\phi(q,\tau)\phi(k-p-q)\right]$

and by iteration $\phi = \int G\eta - \frac{\lambda}{3!}\int\!\!\int\!\!\int\!\!\int G(G\eta)(G\eta)(G\eta) + \dots$

that is



and for instance



NSPT: INTEGRATE on a COMPUTER ...!

NSPT for Lattice Gauge Theories

STARTING POINT: Langevin equation for Wilson Action*

(Batrouni, Katz, Kronfeld, Lepage, Svetisky, Wilson '85)

$$\left(^* S_G = -\frac{\beta}{2N}\sum_P \mathrm{tr}\,(U_p + U_p^\dagger)\right) \qquad (\Diamond)\ \boxed{U_\mu(x;\tau+1) = e^{-F[U(\tau),\eta]}\, U_\mu(x;\tau)}$$

where $F[U(\tau),\eta] = \sum_i T^i F_i \qquad \left([T^i, T^j] = -f_{ijk} T^k\right)$

$$F_i[U(\tau),\eta] = \varepsilon \nabla^i_{x,\mu} S[U] + \sqrt{\varepsilon}\,\eta^i$$

$\eta^i \rightarrow$ GAUSSIAN NOISE

$\nabla^i_{x,\mu}$: GROUP DERIVATIVE

i.e. $f(e^{\alpha\cdot T}U) = f(U) + \alpha^i \nabla^i f(U) + O(\alpha^2)$

$$\sum_i T^i \nabla^i_{x,\mu} S[U] = \frac{\beta}{4N} \sum_{U_p \ni U_\mu(x)} (U_p - U_p^\dagger)_{\text{traceless}}$$

$\rightarrow$ pretty local, as expected

($\Diamond$ actually from $\boxed{\frac{d}{d\tau} U = [-i\nabla S[U] - i\eta]\, U \ldots}$)

$\boxed{\text{NSPT}}$ for LGT (Di Renzo, Marenzoni, Marchesini, Onofri '94)

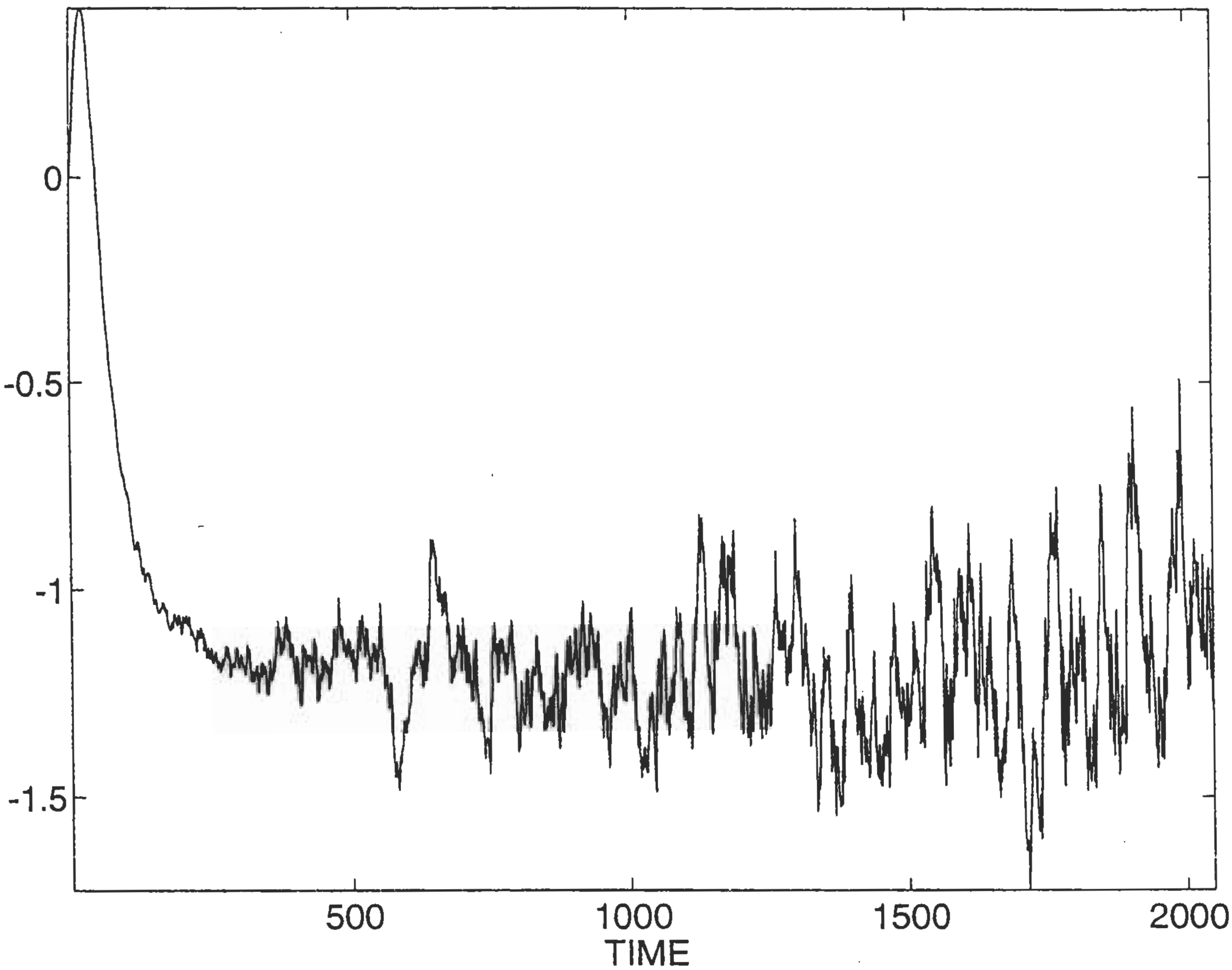
- $\left\{\begin{array}{l} \bullet \text{ either think of} \\ \bullet \text{ or directy write} \end{array}\right.$ $\boxed{\begin{array}{l} U_\mu(x) = \exp\left[A_\mu(x) = \sum_{k>0} \beta^{-k/2} A^{(k)}_\mu(x)\right] \\ U_\mu(x) = 1\!\!1 + \sum_{k>0} \beta^{-k/2}\, U^{(k)}_\mu(x) \end{array}}$

and $\Diamond \longrightarrow$ HIERARCHY of EQUATIONS ... i.e. $F[U] = \sum_{k>0} \beta^{-k/2} F^{(k)}$

BEWARE! Whichever your taste is about • ... you have now DECOMPACTIFIED the formulation $\rightarrow$ divergencies in the non-gauge-invariant sector show up $\rightarrow$ STOCHASTIC GAUGE FIXING...

The divergencies 'round the corner...

$O(\beta^{-2})$ for the 4-d SU(3) plaquette



As a matter of fact, U(1) is enough to understand

$$\dot{A}_\mu(K,\tau) = -K^2\, T_{\mu\nu}(K)\, A_\nu(K,\tau) + \eta_\mu(K,\tau)$$

i.e. $T_{\mu\nu}\dot{A}_\nu = -K^2 T_{\mu\nu} A_\nu + T_{\mu\nu}\eta_\nu$ $\quad \boxed{L_{\mu\nu}\dot{A}_\nu = L_{\mu\nu}\eta_\nu}$

NO RESTORING FORCE
i.e. RANDOM WALK...

Only Gauge Inv. quantities have got a $\lim_{\tau\to\infty}$ in STOCH. QUANTIZ!

STOCHASTIC GAUGE FIXING

Zwanziger '81

$$\dot{A}^a_\mu = -\frac{\delta S[A]}{\delta A^a_\mu} - \boxed{D^{ab}_\mu \nu^b[A]} + \eta^a_\mu$$

i.e. an extra drift...

Since for a GAUGE INVARIANT functional $F[A]$: $D^{ab}_\mu \frac{\delta F[A]}{\delta A^b_\mu} = 0$

$\frac{d}{d\tau} F[A] = \int dx \, \frac{\delta F[A]}{\delta A^a_\mu} \frac{d}{d\tau} A^a_\mu$ is UNAFFECTED

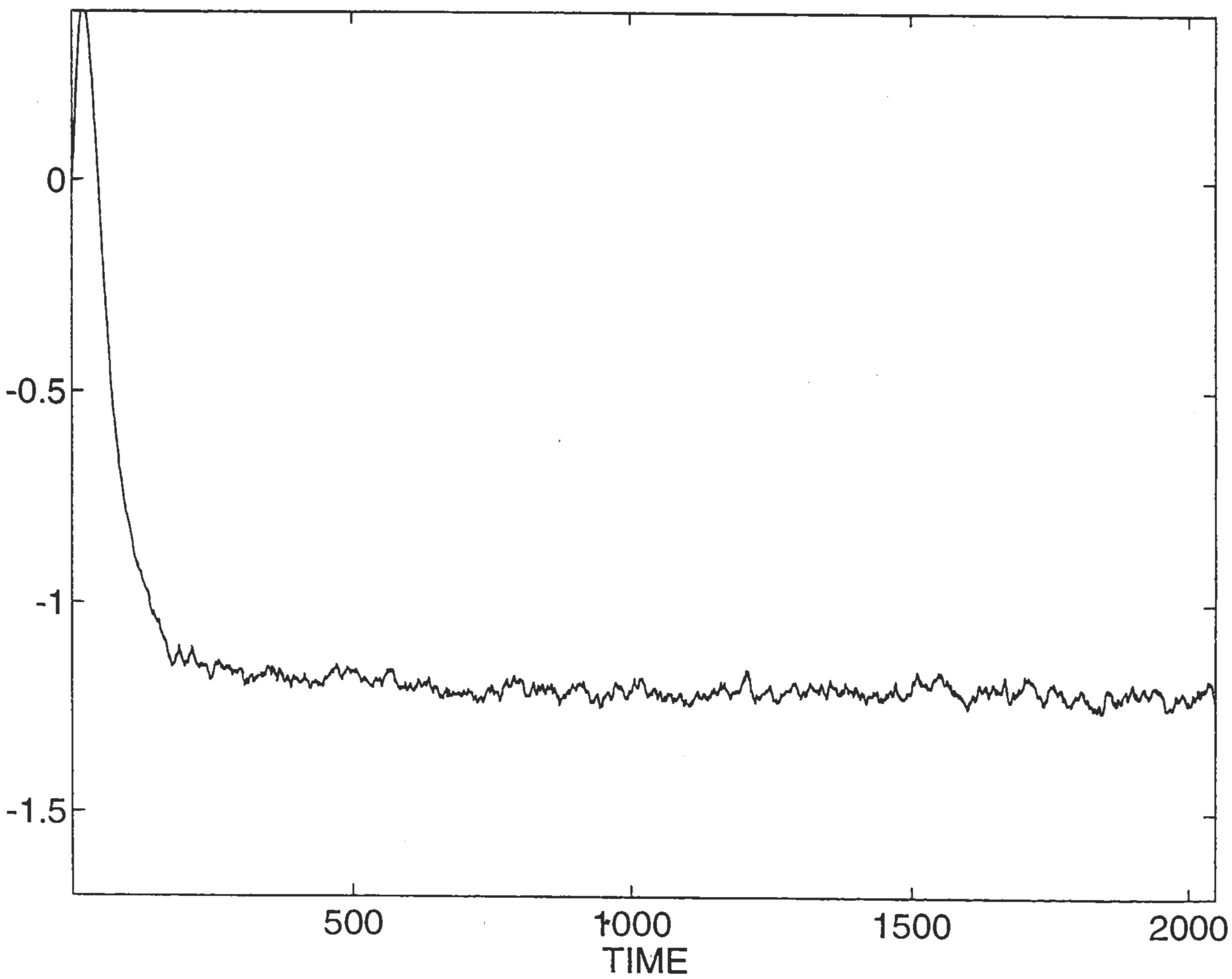
Zwanziger $\boxed{-D^{ab}_\mu \nu^b = \frac{1}{\alpha} D^{ab}_\mu \partial_\nu A^b_\nu}$ $\rightarrow$ the system is attracted by the Gribov region Ω

$$\left\{ \begin{array}{l} \partial_\mu A_\mu = 0 \\ L(A) = -\partial_\mu D_\mu(A) \geq 0 \end{array} \right\}$$

On the LATTICE Rossi Davies Lepage '88

A_μ NORM MINIMISING...

$$U^{(\tau)} \rightarrow U' = e^{-F[U,\eta]} U \longrightarrow U^{(\tau+1)}_\mu(n) = G(n) \, U'_\mu(n) \, G^\dagger(n+\mu)$$



The same $O(\beta^{-2})$ 4-d SU(3) plaquette with SGF...

FERMIONIC LOOPS contribution!

- It works pretty well... e.g. $W = 1 - \frac{1}{3} \mathrm{Tr}\, U_p = \sum_{n=1}^{\boxed{10}} c_n \alpha^n + O(\alpha^{11})$ → Di Renzo & Scorzato '01

  I.R. renormalon behaviour and so on...
  vs POWER-LAW (Rakow & al) ...
  ↳ they now claim $\boxed{\alpha^{16}}$

MAIN LIMITATION till now... QUENCHED APPROXIMATION

i.e. whichever  you want ... but NO FERMIONIC

LOOP  CONTRIBUTION...

- ADDING FERMIONS means FACING "det( )-class" ACTIONS

  (M = fermions, F.P.) $$e^{-S} \det M = e^{-(S - \mathrm{tr} \ln M)}$$

  (Burgio, Di Renzo, Marchesini, Onofri, Pepe, Scorzato '99)

- A SOLUTION for the standard NON-PERTURBATIVE Langevin equation has been known for a long time...

  again Batrouni et al '85

  In principle you shoud simply $S_G \to S_G + S_F = S_G - \mathrm{tr} \ln M$

  and in the equation $\nabla S_G \to \nabla S_G - \nabla \mathrm{tr} \ln M$

  Now... $\nabla^i_{x,\mu} (\mathrm{tr} \ln M) = \mathrm{tr} \left(M^{-1} \nabla^i_{x,\mu} M \right)$

  ↓

  As expected, you face an INVERSE...
  that is NON LOCALITY...

In order to understand Batrouni et al's way out, it is worthwhile to understand a bit better...

Why does Langevin equation work at all?

- generically $\phi(\tau+1) = \phi(\tau) - \varepsilon \frac{\delta S}{\delta \phi} - \sqrt{\varepsilon}\,\eta \equiv \phi(\tau) - f[\phi, \eta]$

Think of it in terms of a PROBABILITY DENSITY $P[\phi, \tau]$

$$P[\phi, \tau+1] = \left\langle \int d\phi \; P[\phi,\tau] \prod_i \delta(\phi_i(\tau+1) - \phi_i(\tau) - f_i) \right\rangle_\eta$$

which means (Taylor expand δ and integrate by parts...)

$$* \quad P[\phi,\tau+1] - P[\phi,\tau] = \sum_m \sum_{i_1 \dots i_m} \frac{\delta}{\delta \phi_{i_1}} \cdots \frac{\delta}{\delta \phi_{i_m}} \frac{1}{m!} \langle f_{i_1} \cdots f_{i_m} \rangle_\eta \, P[\phi, \tau]$$

FOKKER-PLANCK equation

To $O(\varepsilon)$ $\quad P[\phi, \tau=\infty]_{\text{equil.}} : -\frac{\delta}{\delta\phi}\left[\frac{\delta P}{\delta \phi} + \frac{\delta S}{\delta \phi} P\right] = 0 \Rightarrow P \sim e^{-S}$

Batrouni et al '85 $\quad$ Simulate $\quad U(\tau+1) = e^{-F_i T^i}\, U(\tau)$

$$F_i = \varepsilon \left[\nabla^i_{x,\mu} S_G - \mathrm{Re}\left[\xi^\dagger_k M^{-1}_{kl} (\nabla^i_{x,\mu} M)_{lm} \xi_m \right] \right] + \sqrt{\varepsilon}\,\eta_i$$

with an extra random noise ξ_k (k = multi-index) $\langle \xi_i \xi_j \rangle_\xi = \delta_{ij}$

Since what now matters in Fokker Planck (*) is $\langle F_i \rangle_{\xi,\eta}$

and $\langle F_i \rangle_\xi = \varepsilon \left[\nabla^i_{x,\mu} S_G - \mathrm{Tr}\,(M^{-1} \nabla^i_{x,\mu} M) \right] + \sqrt{\varepsilon}\,\eta_i$

$$= \varepsilon\, \nabla^i_{x,\mu} \left[S_G - \mathrm{Tr}\, \ln M \right] + \sqrt{\varepsilon}\,\eta_i$$

which yields $\quad P \sim e^{-(S_G - \mathrm{Tr}\ln M)} = e^{-S_G} \det M$

Of course LIFE IS NOW EASIER... SIMPLY SOLVE $M\psi_i = \xi_i$

in terms of which UPDATING is LOCAL $F_i = \varepsilon\left[\nabla^i_{x,\mu} S_G - \mathrm{Re}(\psi^\dagger \nabla^i_{x,\mu} M \xi)\right] + \sqrt{\varepsilon}\,\eta_i$

- For NSPT LIFE is even EASIER...
  - We first (LAT98) introduced the mecchanism for FADDEEV-POPOV COVARIANT GAUGES...

$$Z^{-1}\int DU\, e^{-(S_G + S_{gf})}\,\Delta_{FP}[U] \ldots \qquad \Delta_{FP}[U] \equiv \det(-\partial_\mu \hat{D}_\mu[U])$$

determined by the response to a GAUGE TRANSFORM.

  - ... but of course now we are FACING $\det M$

$$M_{y\beta b,z\gamma c} = (m+4)\,\delta_{yz}\,\delta_{\beta\gamma}\,\delta_{bc} - \frac{1}{2}\sum_{\nu=\pm 1}^{\pm 4} \delta_{y,z+\nu}\left[1+\gamma_\nu\right]_{\beta\gamma} U_\nu(z)_{bc}$$

$r=1$ WILSON FERMION ACTION

Now EVERYTHING should be EXPANDED in

$$U(\tau+1) = e^{-F}\, U(\tau)$$

$$F_i = \varepsilon\left(\nabla^i_{x,\mu} S_G - \mathrm{Re}\left(\xi_\ell (\nabla^i_{x,\mu} M)_{\ell m} M^{-1}_{mk} \xi_k\right)\right) + \sqrt{\varepsilon}\,\eta_i$$

which means... ALSO $M^{-1} \doteq M^{(0)-1} + \sum_{k>0} \beta^{-k/2} M^{-1\,(k)}$

WHY IT IS FEASIBLE

① EVERYTHING is LOCAL in terms of

$$\xi_k \underbrace{(\nabla^a_{x,\mu} M)^{(i)}_{k\ell}}_{\text{order-}i} \underbrace{\xi^{(j)}_\ell}_{\text{from (!)}}$$

$$(\nabla^a_{x,\mu} M)_{y\beta b, z\gamma c} = -\frac{1}{2}\Big(\delta_{zx}\,\delta_{y,z+\mu}(1+\gamma_\mu)_{\beta\gamma}(T^a U_\mu(x))_{bc} + \ldots$$
$$\ldots - \delta_{z-\mu,x}\,\delta_{y,z+\mu}(1-\gamma_\mu)_{\beta\gamma}(U^\dagger_\mu(x) T^a)_{bc}\Big)$$

(!) $\xi^{(j)}_\ell \equiv M^{-1\,(j)}_{\ell s}\, \xi_s$

② It is trivial that
$$M = M^{(0)} + \sum_{k>0} \beta^{-k/2} M^{(k)}$$
$$\Downarrow$$
$$M^{-1} = M^{(0)-1} + \sum_{k>0} \beta^{-k/2} M^{-1\,(k)}$$

$$M^{-1\,(1)} = -M^{(0)-1} M^{(1)} M^{(0)-1}$$

$$M^{-1\,(2)} = -M^{(0)-1} M^{(2)} M^{(0)-1} - M^{(0)-1} M^{(1)} M^{-1(1)}$$

$$M^{-1\,(3)} = -M^{(0)-1} M^{(3)} M^{(0)-1} - M^{(0)-1} M^{(2)} M^{-1(1)} - M^{(0)-1} M^{(1)} M^{-1(2)}$$

... i.e. 1) only 1 real inverse and 2) an easy recursive relation...

so that
$$\xi^{(0)} \equiv M^{(0)-1}\xi$$
$$M^{-1(1)}\xi \equiv \xi^{(1)} = -M^{(0)-1} M^{(1)} \xi^{(0)}$$
$$M^{-1(2)}\xi \equiv \xi^{(2)} = -M^{(0)-1}\left[M^{(2)}\xi^{(0)} + M^{(1)}\xi^{(1)}\right]$$
...

③ $M^{(0)-1}$ IS DIAGONAL in FOURIER SPACE !

$$M^{(0)-1}(p) \equiv \Delta_p = \frac{m + \frac{1}{2}\hat{p}^2 - i\gamma\cdot\bar{p}}{(m+\frac{1}{2}\hat{p}^2)^2 + \bar{p}^2}$$

Via FFT you go back and forth from Fourier space...
($V \log V$ vs V^2...)

Lippert, Schilling, Toschi, Trentmann, Tripiccione '97

FFT on APE $\longrightarrow$ 1-d (local) FFT + transposition

- 2-d example on a systolic ring

$$\begin{array}{cccc}
m_{00} & m_{10} & \cdots & m_{n0} \\
m_{01} & m_{11} & \cdots & m_{n1} \\
m_{02} & m_{12} & \cdots & m_{n2} \\
\vdots & \vdots & \cdots & \vdots \\
m_{0n} & m_{1n} & \cdots & m_{nn} \\
\uparrow & \uparrow & & \uparrow \\
\text{FFT} & \text{FFT} & & \text{FFT} \\
\text{1-d} & \text{1-d} & & \text{1-d}
\end{array}$$

then transpose...

$$\begin{array}{cccc}
m_{00} & m_{01} & \cdots & m_{0n} \\
m_{10} & m_{11} & \cdots & m_{1n} \\
m_{20} & m_{21} & \cdots & m_{2n} \\
\vdots & \vdots & \cdots & \vdots \\
m_{n0} & m_{n1} & \cdots & m_{nn} \\
\uparrow & \uparrow & & \uparrow \\
\text{FFT} & \text{FFT} & & \text{FFT} \\
\text{1-d} & \text{1-d} & & \text{1-d}
\end{array}$$

and again ...

and then "transpose back"...

easier on APEmille...

- SOME DETAILS on IMPLEMENTATION on APEmille

(and now also on PC's ...)

- FFT from $\boxed{\text{1-d FFT + transposition}}$ Lippert-Schilling-Toschi-Trentman-Tripiccione '97

↓

this is where you need LOCAL ADDRESSING...

1. PRE-SKEWING 2. ROW-SHIFTING 3. RE-SKEWING

and so on...

- SINGLE/DOUBLE PRECISION

- TIMING (seconds/iteration × # of processors)

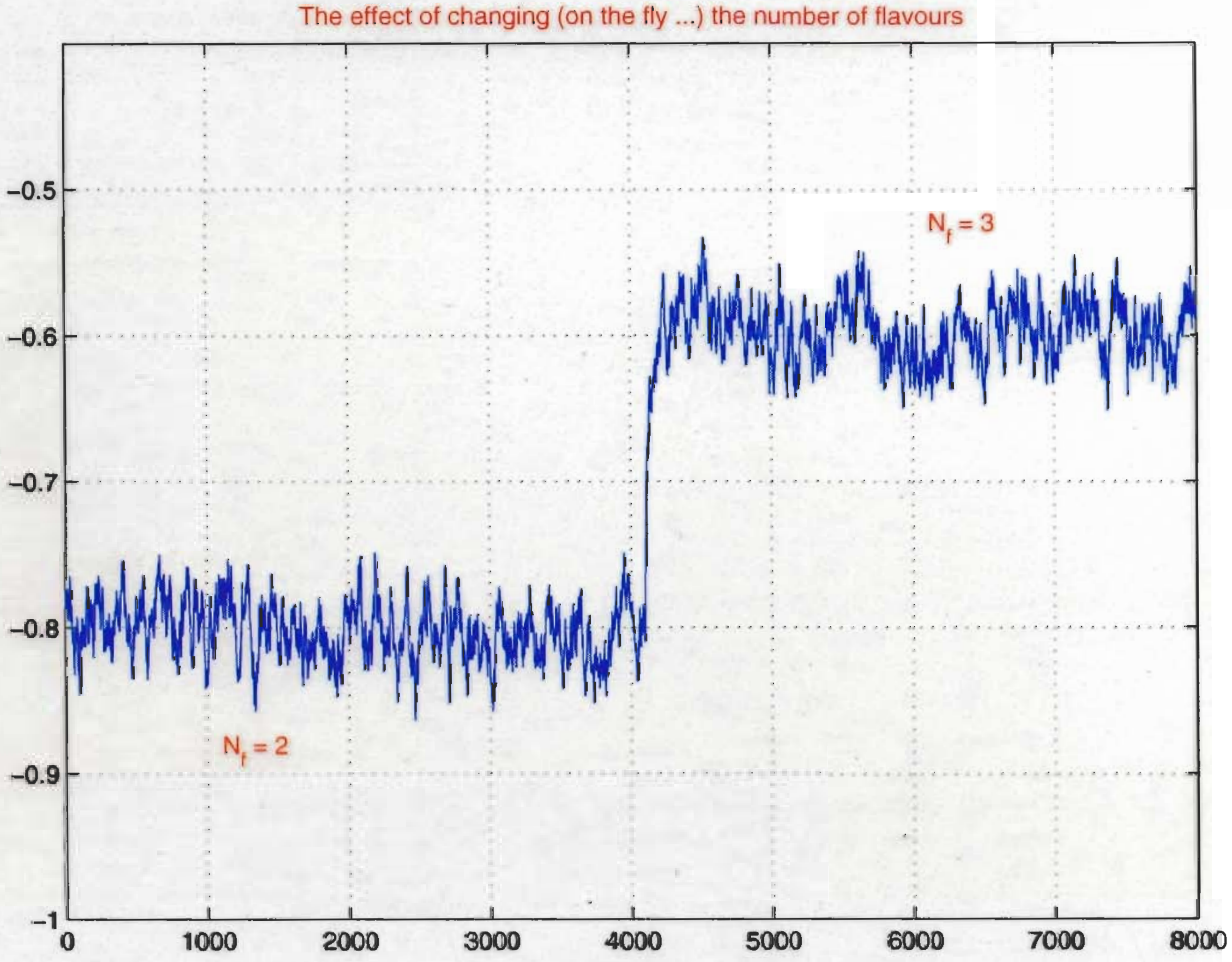
	Lattice size	order		$n_f=0$		$n_f=2$	
on a BOARD	8	g^6	ORDER DEPEND. $\sim \frac{O(O-1)}{2}$	28	↓	48	UNQUENC. Overhead $\sim \frac{5}{3}$
	8	g^8		49	idem	82	
	8	g^{10}		79	↓	129	
on a UNIT	16	g^6		453		814	
on a CRATE	32	g^6		7168		12979	

- AUTOCORRELATION

(basic plaquette, L = 32)

n_f	ε	β^{-1}	β^{-2}	β^{-3}
0	0.01	~50	~70	~100
2	0.005	~100	~120	~150

Easy to fit N_f DEPENDENCE



m_b from the LATTICE

Despite the fact that one can not accomodate the b-QUARK on the LATTICE, its mass determination is one of the most brilliant result from LATTICE GAUGE THEORIES...

I am aware of at least 4 strategies

* NRQCD — S. Collins and others (2002 - Contin. 2000)
* HQET + Perturbation Theory — G. Martinelli, C. Sachrajda et al. (Martinelli, Gimenez, Giusti, Rapuano 2000) (*)
* HQET in a Non Perturbative framework — Alpha Collaboration (R. Sommer et al, 2004)
* Step-scaling method — Roma TV (Petronzio et al, 2003)

(*) + Di Renzo, Scorzato 2004 → $\boxed{\overline{m}_b^{(UNQ)}(\overline{m}_b) = 4.21 \pm 0.03 \pm \underline{0.04}\ \text{GeV}}$

$\underline{0.04}$ was halved by our computation

THE RESIDUAL MASS (δm) in LHQET to α^3 order

* The THEORETICAL FRAMEWORK (a sketch of) Martinelli & Sachrajda et al

$\mathcal{L}_{HQET} = \bar{h} D_4 h$ looks simple, but contains many subtleties...

... many of which have to do with the definition of the heavy quark mass m_Q itself $\quad P_Q^\mu = m_Q v^\mu + K^\mu \quad\quad K^\mu \sim \Lambda_{QCD}$

The MAIN POINT : the $\frac{1}{m_Q}$ expansion is not well defined...

AMBIGUITIES show up

dim REGUL.
- UV renormalons in the MATRIX ELEMENTS of HQET
- IR renormalons in the COEFF. FUNCTIONS that match to QCD (effective theory)

LATTICE REGUL.
- non perturbative POWER DIVERGENCES

The ambiguities in the coefficient functions are expected to cancel with those in the HQET matrix elements.

One needs to face the problem of how to define the expansion parameter itself (m_Q) in a way free of ambiguities ...

being aware that

(!) m_{POLE} is not good with this respect (Beneke & Braun '94)

The SUMMARY of a (LONG) STORY:

You can not simply go via $M_B = m_b + \mathcal{E} + O(1/m_b)$

M_B: mass of a physical hadron

m_b: (heavy quark) HQET mass parameter

$\mathcal{E}$: binding energy (DIVERGENT!)

We follow Martinelli et al

(a) MATCH QCD PROPAGATOR to LHQET PROPAGATOR: you get

$$m_b^{POLE} = M_B - \mathcal{E} + \underbrace{\delta m}_{\text{RESIDUAL MASS}} + O(1/m_b)$$

a linearly divergent mass counterterm

(b) In P.T.

$$\delta m = \sum_{n \geq 0} \overline{X}_n \, \alpha_0^{n+1}$$

(c) In P.T., you can go to $\overline{MS}$

$$\overline{m}_b(\overline{m}_b) = m_b^{POLE} \left[1 + \sum_{n \geq 0} \left(\frac{\alpha_s(\overline{m}_b)}{\pi} \right)^{n+1} D_n \right]$$

D_n from Chetyrkin et al, Melnikov et al

(d) Put things togheter

$$\boxed{\overline{m}_b(\overline{m}_b) = \left[M_B - \mathcal{E} + \sum_n (\alpha_s(\overline{m}_b))^{n+1} \frac{X_n}{a} \right] \left[1 + \sum_n \left(\frac{\alpha_s(\overline{m}_b)}{\pi} \right)^{n+1} D_n \right]}$$

in which

1. δm has to cancel the LIN. DIV. from $\mathcal{E}$ ...
2. ... and a RENORMALON ambiguity as well (m_b^{POLE}!)
3. Everything takes place in P.T.! → You have to stick to a FIXED ORDER!
4. You need X_2 ...

The STRATEGY

Di Renzo, Scorzato 2001, 2004

- For a WILSON LOOP $\langle W \rangle = \exp\left(-c\, L/a\right) W_{log}$

  linear diverg. **our goal**

  log divergencies: - coupling - corners...

- Compute

$$V(R) \equiv \lim_{T\to\infty} V_T(R)$$

$$V_T(R) \equiv \log\left(\frac{W(R,T-1)}{W(R,T)}\right)$$

in NSPT... (T big...)

- Corner div's disappear and as for the coupling

$$\boxed{V(R) = 2\,\delta m + V_{Coul}(R) \equiv 2\,\delta m - C_F\,\frac{\alpha_V(R)}{R}}$$

- You Know that

$$V(R) = 2\,\delta m - \frac{C_F}{R}\left(\alpha_0 + c_1(R)\,\alpha_0^2 + c_2(R)\,\alpha_0^3 + \ldots\right)$$

$$c_1(R) = 2 b_0 \log R + 2 b_0 \log \frac{\Lambda_V}{\Lambda_0}$$

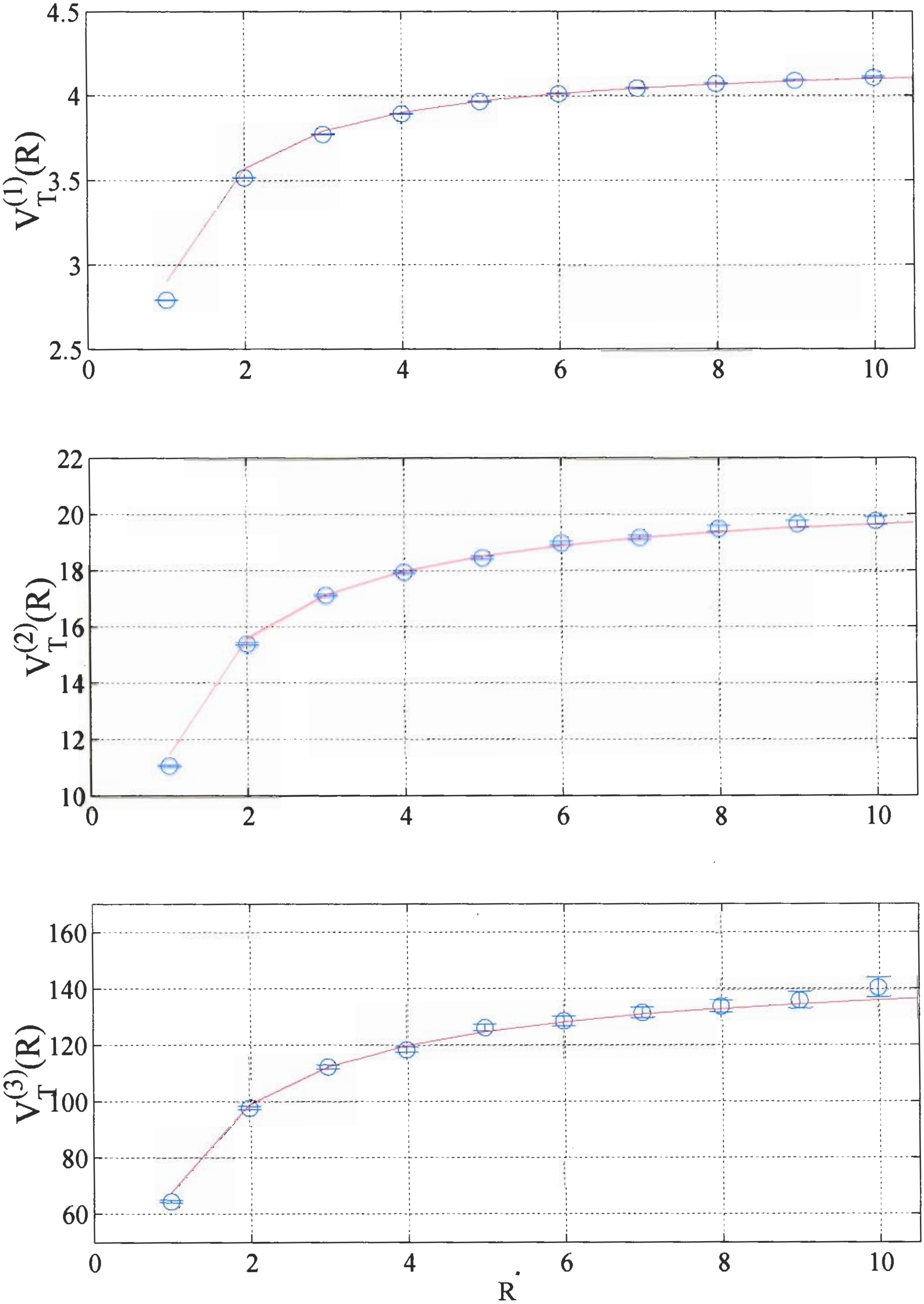
$$c_2(R) = c_1(R)^2 + 2 b_1 \log R + 2 b_1 \log\frac{\Lambda_V}{\Lambda_0} + \frac{b_2^{(V)} - b_2^{(0)}}{b_0}$$

in which everything is KNOWN BUT δm (due to Schröder, Panagopoulos et al)

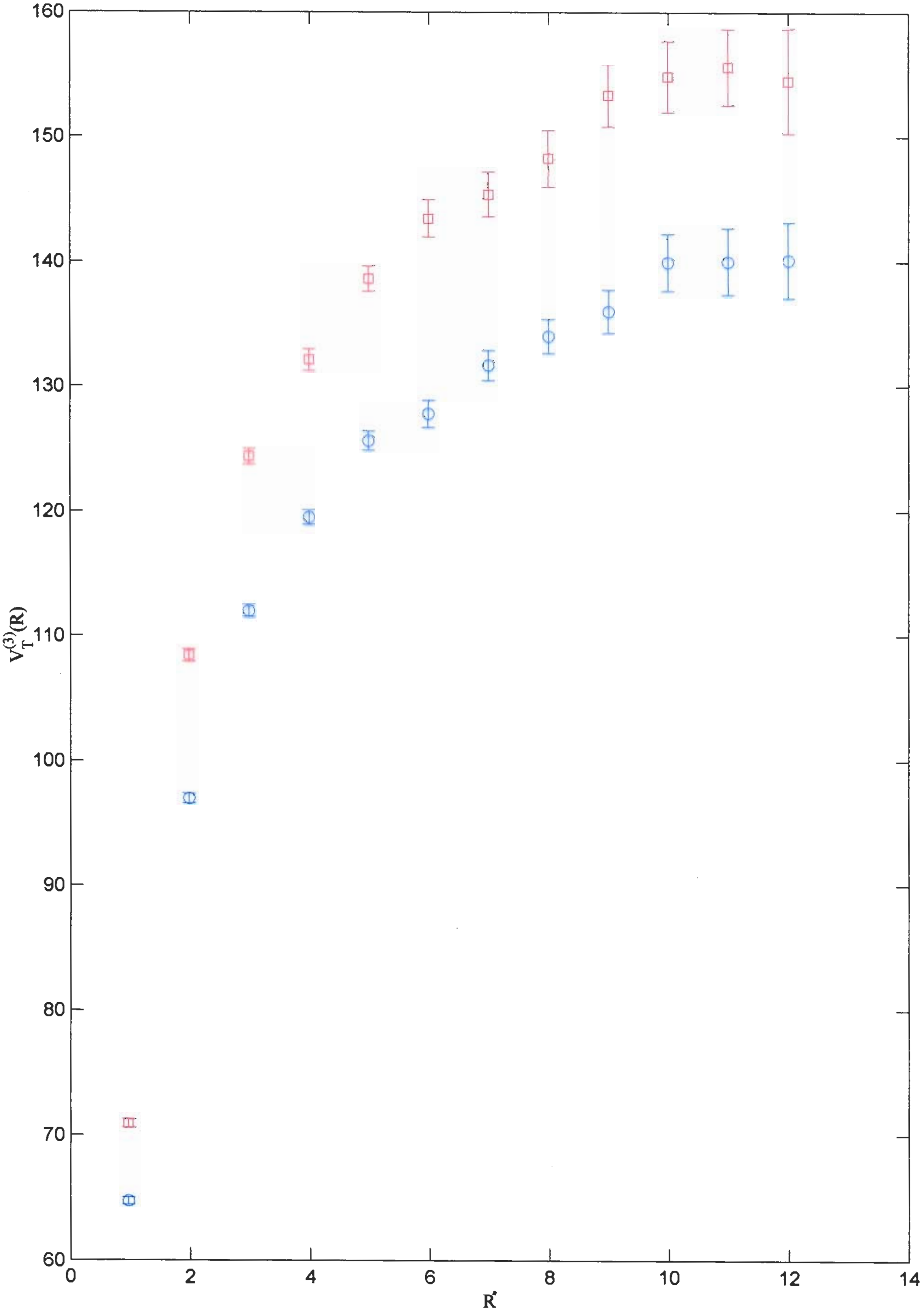
→ **COMPUTE $V(R)$ and FIT δm**

You work in R-T intervals such that $\begin{cases} R > 3 \\ T > 2.5\,\bar{R} \end{cases}$ and choose by χ^2...

$$\bar{X}_2^{(n_f=0)} = 86.2(6) \qquad \bar{X}_2^{(n_f=2)} = 76.7(6)$$



4.5
4
3.5
3
2.5
$V_T^{(1)}(R)$
0
2
4
6
8
10
22
20
18
16
14
12
10
$V_T^{(2)}(R)$
0
2
4
6
8
10
160
140
120
100
80
60
$V_T^{(3)}(R)$
0
2
4
6
8
10
R



160
150
140
130
120
110
100
90
80
70
60
0
2
4
6
8
10
12
14
$V_T^{(3)}(R)$
R'

The QUARK PROPAGATOR

as a prototype for computations in the fermionic sector...

- We want the 2 points VERTEX FUNCTION

$$\Gamma_2(p^2;m) = S(p^2;m)^{-1} = i\not{p} + m - \Sigma(p^2;m) \qquad (a \to 0)$$

actually for $m=0$ $\longrightarrow$ CRITICAL MASS for Wilson fermions
$\hookrightarrow$ Z_q in RI' scheme

$$\Sigma(p^2;m) = \Sigma_c + i\not{p}\,\Sigma_1 + m\,\Sigma_2$$

BEWARE! We always compute at $\boxed{a \neq 0}$ (and on a finite $L=32$)

We want $\langle M^{-1}_{ii} \rangle$ $\Longrightarrow$ compute from (sources...) $A_{ij} = \sum_{k,l} \xi_k^{(i)} A_{kl} \xi_l^{(j)}$

$\xi_k^{(a)} = \delta_{ak}$

(a) COMPUTE $S(pa)$ in LANDAU gauge (take advantage of HYPERCUBIC SYMMETRY!)

(b) INVERT TO GET Γ_2

(c) PROJECT OUT γ-COMPONENTS

PLUG THE CRITICAL MASS COUNTERMS IN!

What do you expect? What do you have to deal with?

$a \neq 0$ → mimic the expansion in terms of hypercubic invariants, which are constructed from power of pa!

(You are taking the CONTINUUM LIMIT...)

$\boxed{\text{CRITICAL MASS}}$ $\Sigma_c(p=0, \delta m_B, g_0) = \delta m_0$

Having plugged counterterms in, first and second order must vanish!

As for third order (expansion in β^{-1}) $\delta m_b^{n_f=2} = 11.79^{(+2)}_{(-5)}$

As expected, POOR CONVERGENCE PROPERTIES...

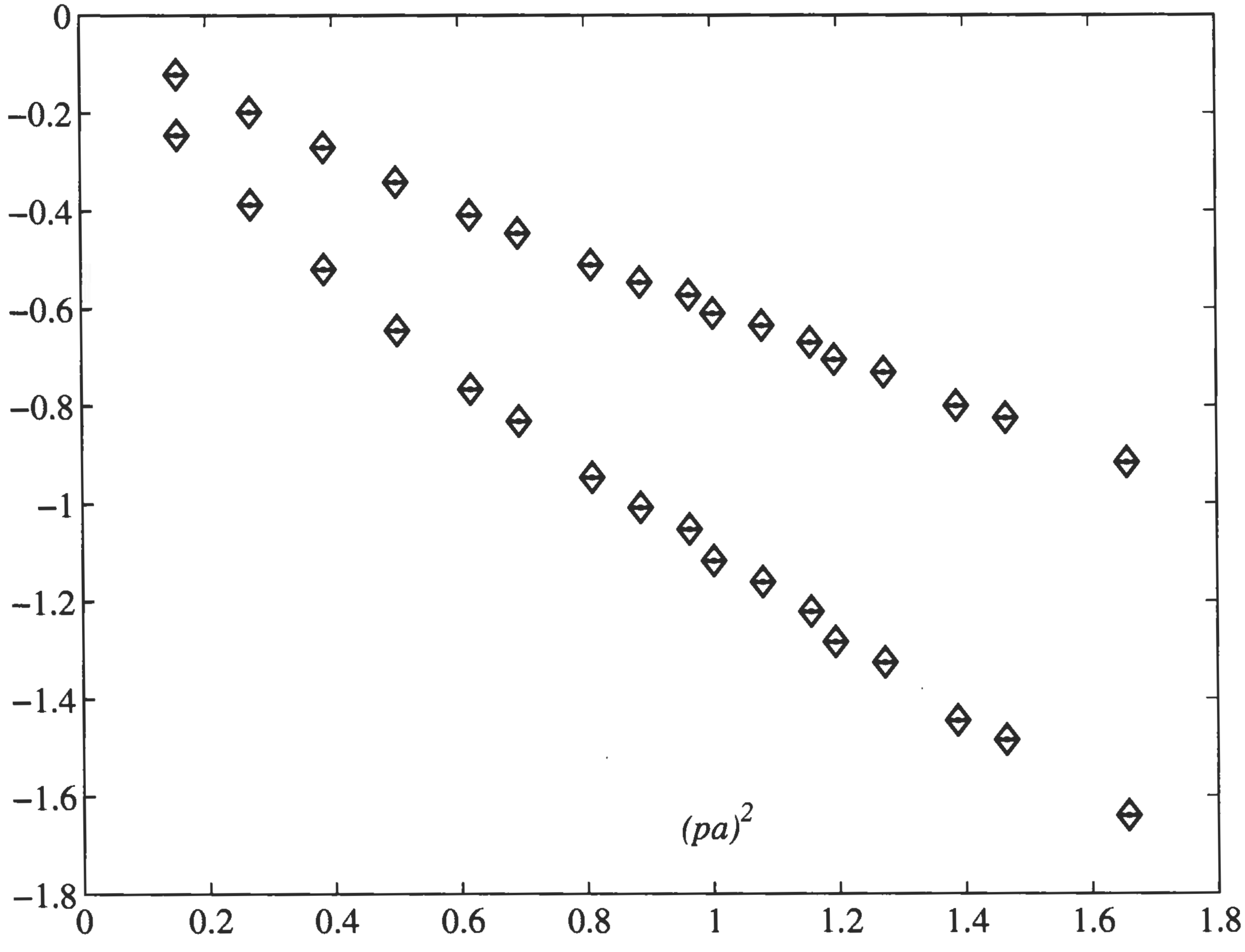
i.e. $\beta = 5.6$

up to β^{-1} (ONE LOOP)	PT/NP	$\sim$	0.55
up to β^{-2}		$\sim$	0.71
up to β^{-3}		$\sim$	0.79

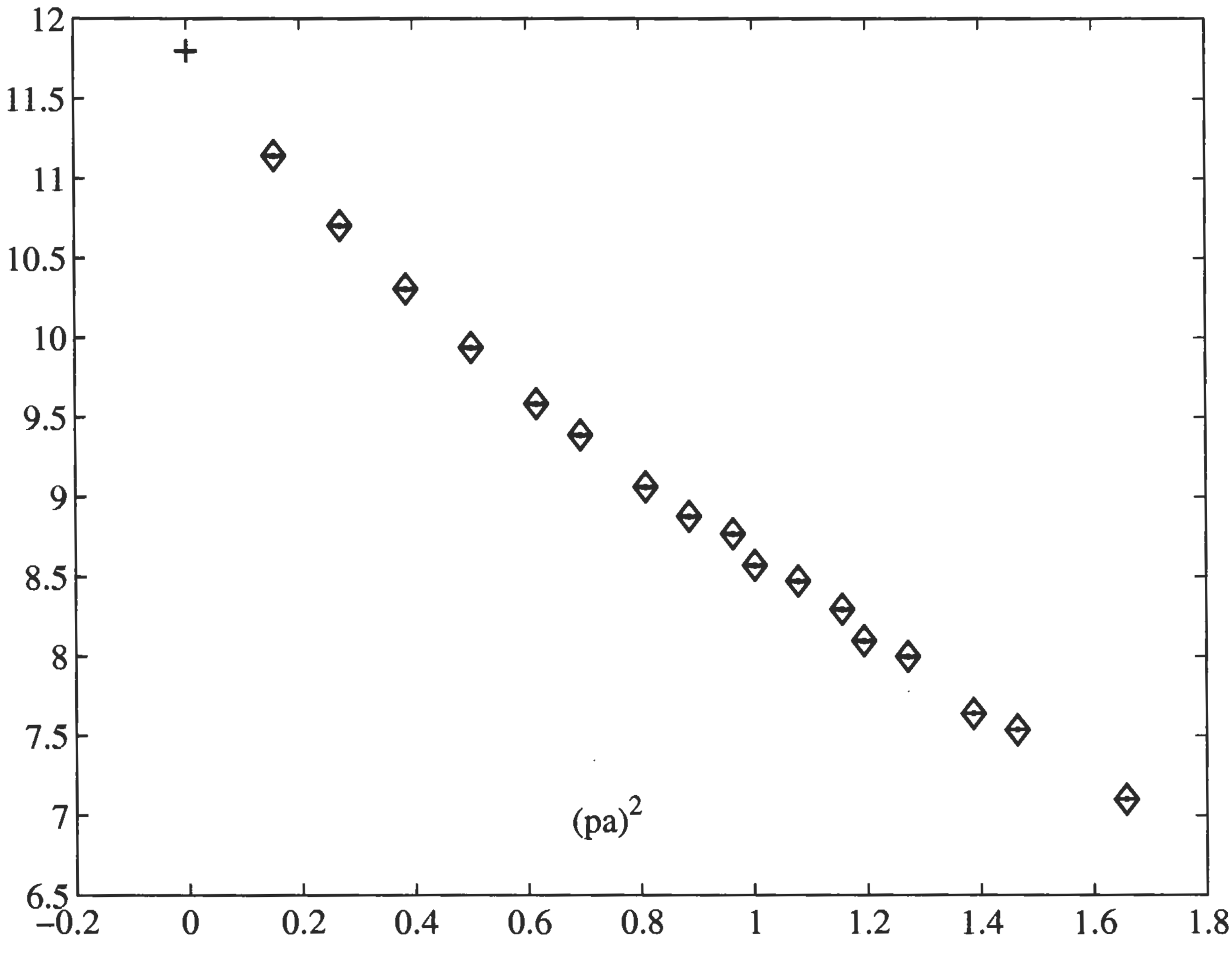
BPT is quite a sterile exercise in this context...

→ GO for Z's! (log div → sound reference point!)

Σ_c (β^{-1} and β^{-2})



Σ_c (third loop)



Z_q in RI'

Compute from $$Z_q' = -i\,\frac{1}{12}\,\frac{\mathrm{Tr}(\not{p}\, S^{-1})}{p^2}$$

KEY POINT: You Know (J. Gracey 2003) the ANOMALOUS DIMENSION!

(This is true for all the bilinears! CURRENTS are on their way...)

We compute $$\boxed{\frac{\mathrm{Tr}\, \gamma_\mu S^{-1}(p)}{p_\mu}}$$

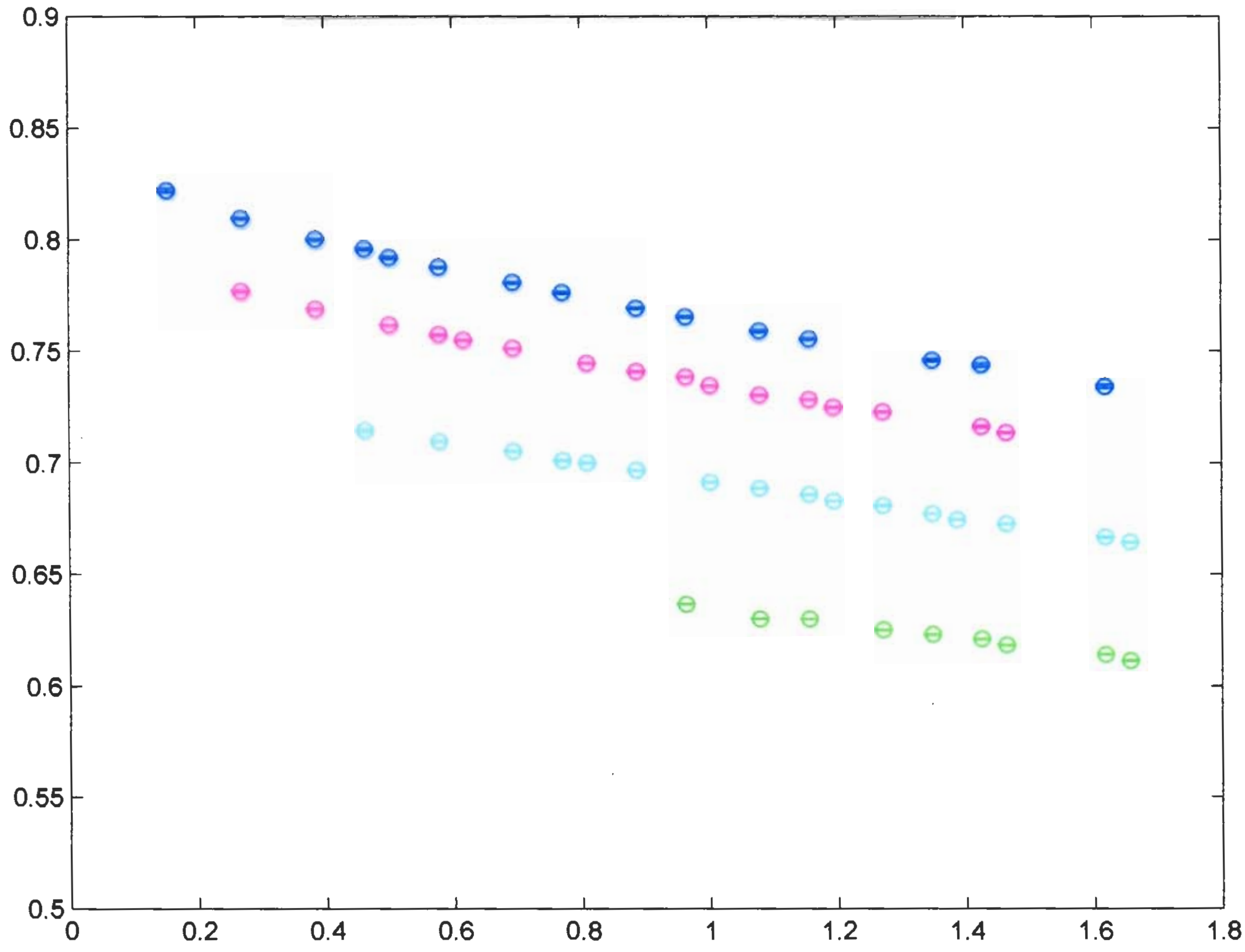
Beware! This depends on $|p_\mu|$ (again $a \neq 0$ ...)

$\downarrow$

Expect "FAMILIES" of curves...

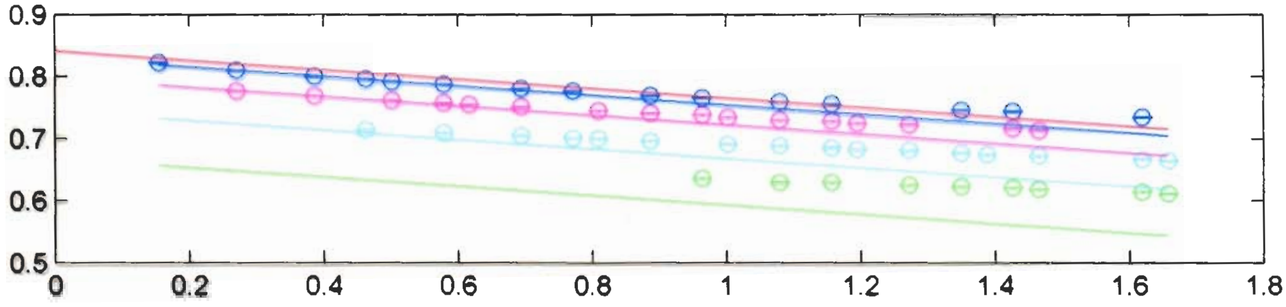
FIT Σ_1 accordingly...

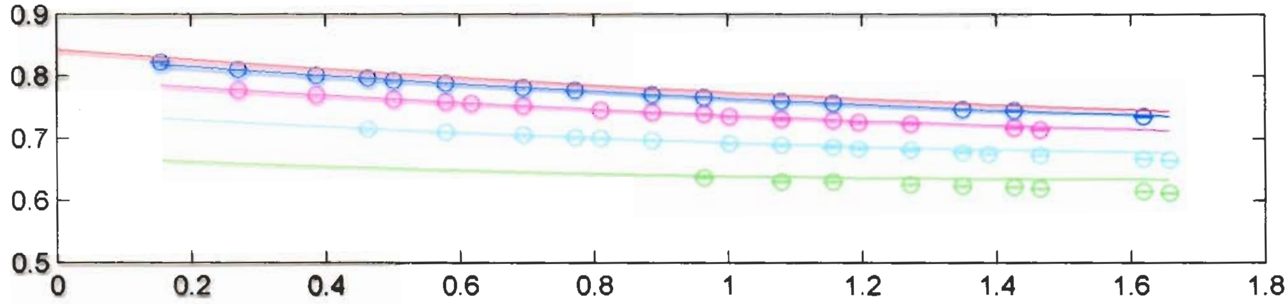
SIMPLE NOTE: at tree level $\not{p} = \gamma^\mu p^\mu (1 + O(p^{\mu 2}))$ ...

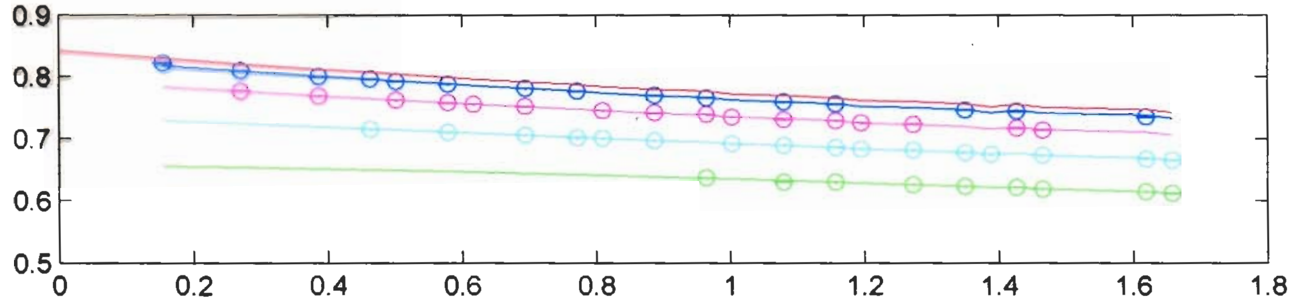

0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8

ONE LOOP

(Landau gauge → NO LOG)







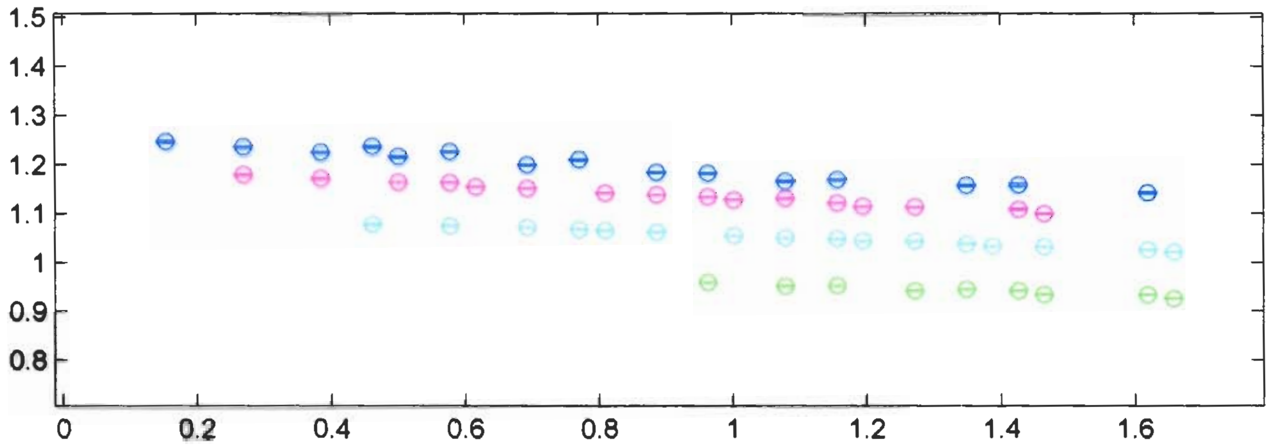
$$-Z_q'^{(1)} = 0.843(1)$$

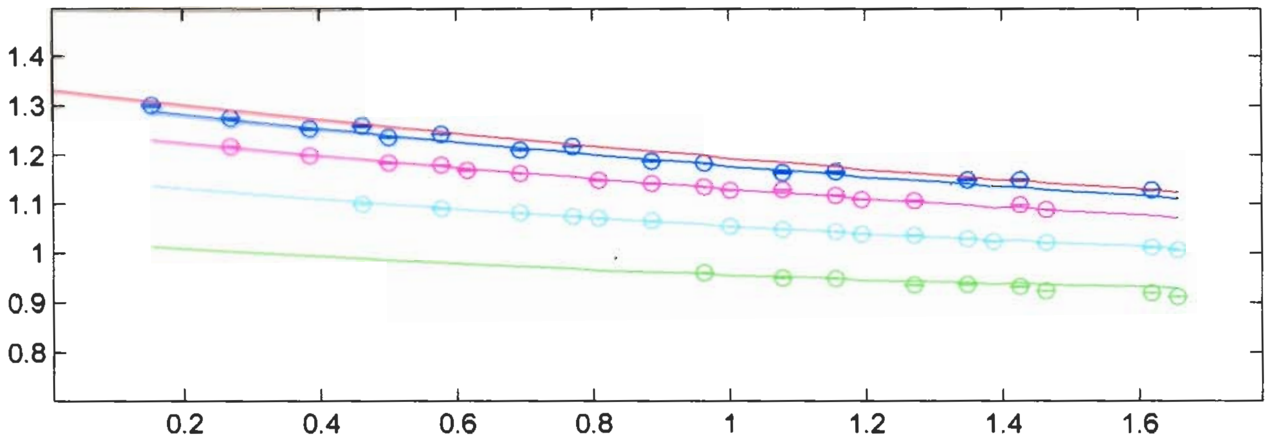
TWO LOOP:

LANDAU GAUGE $\longrightarrow$ LOG, but NO LOG^2

$\downarrow$

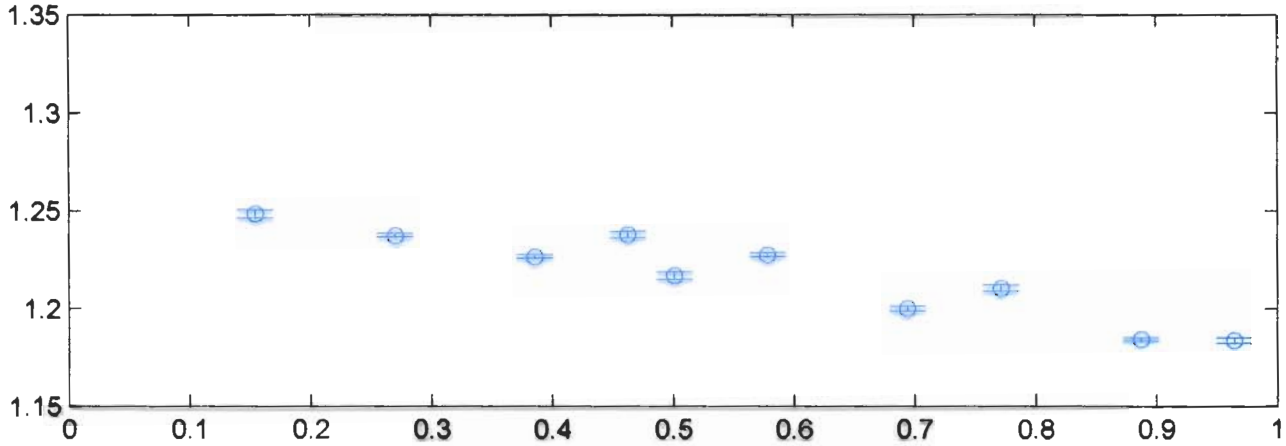
TAKE IT INTO ACCOUNT

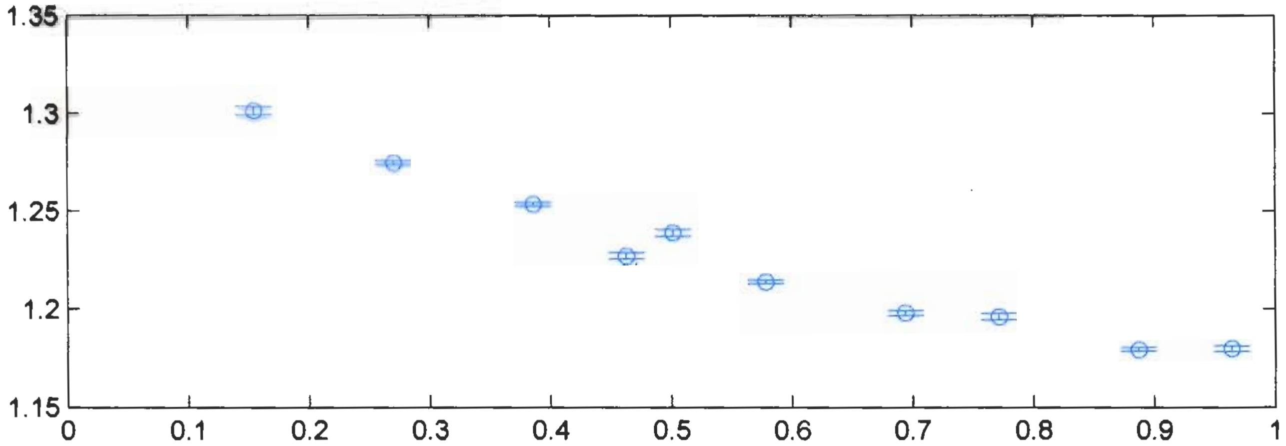




$-Z'_q = 1.33(2)$

$(a\mu = 1)$.


1.35
1.3
1.25
1.2
1.15
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


1.35
1.3
1.25
1.2
1.15
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

CONCLUSIONS

- IT WORKS PRETTY WELL...
- DATABASE of CONFIGURATIONS READY
- CURRENTS ON THEIR WAY
- STILL A LOT OF WORK TO DO:

  4-fermions operator (TM-QCD)

  OVERLAP (1 LOOP OK!)

  IMPROVEMENT-COEFFICIENTS ?

  ...