2.4 Quantization, path integral (remarks only)

- so far, have seen non-Abelian gauge symmetry at work $\Rightarrow \mathcal{L}_{QCD}$.
  now, work out consequences for particle physics interactions
  → need rules for computing Feynman diagrams
  → apply rules to compute amplitudes, cross sections

- local gauge symmetry $\Rightarrow$ some Lagrangian dof's are unphysical
  ($\hat{=}$ can be adjusted arbitrarily by gauge trafos)

cf. QED: in functional integral $\int \mathcal{D}A \, e^{iS[A]}$ the photon part

was
$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)$$
$$= \frac{1}{2}\int d^4x \, A_\mu(x) \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu\right) A_\nu(x)$$
$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k) \left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right) \tilde{A}_\nu(-k)$$

(FT $A(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{A}(k)$, $\int d^4x \, e^{-ikx} = (2\pi)^4 \delta^{(4)}(k)$)

$\Rightarrow$ for $\tilde{A}_\mu(k) = k_\mu a(k)$, $S = 0$ $\Rightarrow$ $\int \mathcal{D}A \, e^{i0}$ diverges! ↯
(a: arbitrary scalar fct)

(($\Leftrightarrow$ (...) has no inverse: cannot solve $(-k^2 g_{\mu\nu} + k_\mu k_\nu)\tilde{D}_F^{\nu\rho}(k) = i g_\mu{}^\rho$
for Feynman propag $\tilde{D}_F$))

recall Abelian gauge invariance $A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$
$\Rightarrow$ field configurations that are gauge-equivalent to $A_\mu(x) = 0$ did ↯

→ the way out was Faddeev-Popov gauge fixing
[Phys. Lett. 25B (1967) 29]

result: $S \to S + \int d^4x \left(-\frac{1}{2\xi}(\partial_\mu A^\mu)^2\right)$

$\Leftrightarrow$ can solve $\left(-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu\right) \tilde{D}_F^{\nu\rho}(k) = i g_\mu{}^\rho$:

$$\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2 + i\varepsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right) \quad \text{photon propagator}$$

→ propagator depends on arbitrary parameter ξ ? ↯
physics does not: QED vertex is such
that ξ drops out of S-matrix elements
(due to the Ward-Takahashi identities)

⇝ similar structure in QCD; ξ-cancellations more complicated.

- we will make use of functional methods

  → most useful for interacting QFT's:
  path integral method, relying on functional integration

  → for (many) more details: [QFT lecture]
  [Peskin/Schroeder, §9]

- reminder of a functional derivative:

  def. $\delta_{J(x)} J(y) = \delta^{(4)}(x-y)$ or $\delta_{J(x)} \int d^4y\, J(y)\,\phi(y) = \phi(x)$

  ⇒ can take functional derivatives as usual,

  e.g. $\delta_{J(x)}\, e^{i\int d^4y\, J(y)\phi(y)} = i\phi(x)\, e^{i\int d^4y\, J(y)\phi(y)}$

  e.g. $\delta_{J(x)} \int d^4y\, (\partial_\mu J(y))\, A^\mu(y) = -\partial_\mu A^\mu(x)$ ((after partial integration))

- reminder of the generating functional of correlation functions

  $$Z[J] = \int \mathcal{D}\phi\; e^{i\int d^4x\,[\mathcal{L} + \underbrace{J(x)\phi(x)}_{\text{source term}}]}$$

  such that
  $$\langle 0|\overset{\text{time ordering}}{T}\phi(x_1)\phi(x_2)|0\rangle = \frac{\int\mathcal{D}\phi\,\phi(x_1)\phi(x_2)\,e^{i\int d^4x\,\mathcal{L}}}{\int\mathcal{D}\phi\,e^{i\int d^4x\,\mathcal{L}}}$$
  $$= \frac{1}{Z[0]}\left(-i\delta_{J(x_1)}\right)\left(-i\delta_{J(x_2)}\right) Z[J]\Big|_{J=0}$$ very elegant!

- to see the elegance of the $Z[J]$ formulation,
  consider a free scalar theory, $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$

  $$\Rightarrow \int d^4x\,[\mathcal{L}_0 + J\phi] \overset{\text{PI}}{=} \int d^4x\,[\tfrac{1}{2}\phi(\underbrace{-\partial^2 - m^2 + i\varepsilon}_{}) \phi + J\phi]$$
  ↳ convergence factor for functional integral, $\varepsilon > 0$

  complete the square: $\phi \to \phi + i\int d^4y\, D_F(x-y)\,J(y)$

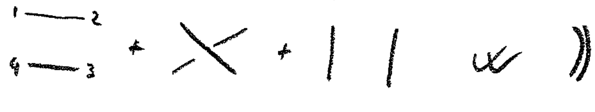
  where $(-\partial^2 - m^2 + i\varepsilon)\, D_F(x-y) = -i\delta^{(4)}(x-y)$, D_F is Greens fct

  $$= \int d^4x\,\left[\mathcal{L}_0 + \frac{i}{2} J(x)\int d^4y\, D_F(x-y)\,J(y)\right]$$

  → $Z_{\text{free}}[J] = Z_{\text{free}}[0]\; e^{-\frac{1}{2}\int d^4x\, d^4y\, J(x) D_F(x-y) J(y)}$

  ⇒ two-point function $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle_{\text{free}} \overset{\text{(check?!)}}{=} \ldots = D_F(x_1 - x_2)$

  ⇒ four-point function $\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle_{\text{free}} = D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}$

  ((where $D_{ij} \equiv D_F(x_i - x_j)$; ))

  ⇒ etc