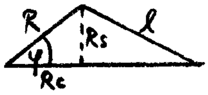
Klausur zu EMTP I (31. März. 2008) (Ergebnisse ab 4. Apr. online) Lösung

① $\vec{r}(t) = R(c\vec{a} + s\vec{b})$ mit $\vec{a} \equiv (\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2})$ und $\vec{b} \equiv (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$\vec{v} = \dot{\vec{r}} = R\omega(-s\vec{a} + c\vec{b})$, $\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} = mR^2\omega\,(c^2+s^2)(\vec{a} \times \vec{b}) = mR^2\omega\,(-\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2})$

② (a) $\partial_x x^x = \partial_x e^{x \ln x} = (\ln(x) + 1)\, x^x$

(b) $\partial_x \frac{sh}{ch} = \frac{ch}{ch} - \frac{sh^2}{ch^2} = 1/ch^2$

③ (a) 

, $x(t) = Rc + \sqrt{l^2 - (Rs)^2}$ (b) $\vec{v}_p = R\omega(-s, c)$

(c) $\vec{v}_p(t = \frac{\pi}{2\omega}) = R\omega(-1, 0)$, $\dot{x}(t) = -R\omega s + \frac{1}{2}\frac{1}{\sqrt{\ }}(-R^2 2sc\omega)$, $\dot{x}(t = \frac{\pi}{2\omega}) = -R\omega$ ✓✓

④ (a) $\frac{1}{a-x} = \frac{1}{a}\frac{1}{1-x/a}$ = geom. Reihe ! $= \frac{1}{a}\sum_{n=0}^{\infty}(\frac{x}{a})^n \Rightarrow \frac{1}{a}(\frac{x}{a})^{16}$

(b) Taylor: $f(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}f^{(n)}(0)$. $f^{(0)}(0) = 1$;

$f'(x) = -2x e^{-x^2}$, $f'(0) = 0$; $f''(x) = (-2 + 4x^2)e^{-x^2}$, $f''(0) = -2$;

$f'''(x) = (4x + 8x - 8x^3)e^{-x^2}$, $f'''(0) = 0$; $f''''(x) = \ldots$, $f''''(0) = 12$;

$\Rightarrow f(x) = 1 + \frac{x^2}{2!}(-2) + \frac{x^4}{4!}12 + \ldots = 1 - x^2 + \frac{1}{2}x^4 + \ldots$

⑤ (a) z.B. Ans. $v(t) = Ae^{-\alpha t} + Be^{\beta t}$; $\dot{v} = -\alpha A e^{-\alpha t} + \beta B e^{\beta t} \stackrel{!}{=} -\alpha A e^{-\alpha t} - \alpha B e^{\beta t} + k_0 e^{\beta t}$

$\Rightarrow B = k_0/(\alpha+\beta)$; $v(0) = A + B \stackrel{!}{=} 0 \Rightarrow v(t) = \frac{k_0}{\alpha+\beta}(e^{\beta t} - e^{-\alpha t})$

((oder z.B. $v = e^{-\alpha t}u$, $\boxed{\dot{u} = k_0 e^{(\alpha+\beta)t}, u(0) = 0}$, $u = \frac{k_0}{\alpha+\beta}(e^{(\alpha+\beta)t} - 1)$))

(b) $v(t) \approx \frac{k_0}{\alpha+\beta}[1 + \beta t + \ldots - (1 - \alpha t + \ldots)] = k_0 t + O(t^2)$

⑥ $-\partial_x V \stackrel{!}{=} \alpha y \Rightarrow V = -\alpha xy + f(y,z)$; $-\partial_y V = \alpha x - \partial_y f \stackrel{!}{=} -x + z^2 \Rightarrow \boxed{\alpha = -1}$, $f(y,z) = -yz^2 + g(z)$

$-\partial_z V = 2yz - \partial_z g \stackrel{!}{=} \beta yz \Rightarrow \boxed{\beta = 2}$, $g(z) = \text{const} \stackrel{\text{z.B.}}{=} 0$; $\Rightarrow V = xy - yz^2$

⑦ Zähler $= 1 + \frac{s^2}{2} + \frac{s^4}{24} + \ldots - 1 = \frac{s^2}{2}(1 + \frac{s^2}{12} + \ldots)$, Nenner $= \ln(1 + \frac{s^2}{2} + \frac{s^4}{24} + \ldots) = \frac{s^2}{2} + \frac{s^4}{24} + \ldots - \frac{1}{2}(\frac{s^4}{4} + \ldots) = \frac{s^2}{2}(1 - \frac{s^2}{6} + \ldots)$

$\Rightarrow f(s) = (1 + \frac{s^2}{12} + \ldots)(1 + \frac{s^2}{6} + \ldots) = 1 + \frac{s^2}{4} + \ldots$, $V = \text{const} + \frac{1}{2}(\frac{k}{2})x^2$, $\omega = \sqrt{\frac{k}{2m}}$

⑧ $H^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbb{1} \Rightarrow H^3 = -H$, $H^4 = \mathbb{1}$, also wie i !

$e^{tH} = \sum_{n=0}^{\infty}\frac{t^n H^n}{n!} = \mathbb{1}\sum_{n=0}^{\infty}\frac{(-t^2)^n}{(2n)!} + H\sum_{n=0}^{\infty}\frac{(-)^n t^{2n+1}}{(2n+1)!} = \mathbb{1}\cos(t) + H\sin(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$

⑨ I ok ; II $0 \stackrel{!}{=} (5-\lambda)^3 + 2\cdot 4^3 - 3\cdot 4^2(5-\lambda) = 13 - 27\lambda + 15\lambda^2 - \lambda^3 = (1-\lambda)(13 - 14\lambda + \lambda^2)$

$= (1-\lambda)(1-\lambda)(13-\lambda) \Rightarrow \lambda_1 = 13$, $\lambda_2 = \lambda_3 = 1$; ($\lambda = 1$ geraten)

III $5 + 5 + 5 = 13 + 1 + 1$ ✓✓

IV λ_1 : $\begin{pmatrix} -8 & 4 & 4 \\ 4 & -8 & 4 \\ 4 & 4 & -8 \end{pmatrix}\begin{pmatrix} r \\ s \\ t \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{f}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $\lambda_{2,3}$: $\begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{pmatrix}\begin{pmatrix} r \\ s \\ t \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow 0 = r + s + t$

Entartung: wähle $\vec{f}_2 \perp \vec{f}_3$, z.B. $\vec{f}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{f}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} (= \vec{f}_1 \times \vec{f}_2)$; V ok.

⑩ (a) $\dot{v}^{(0)} + \dot{v}^{(1)} + \ldots = g(1 - \alpha v^{(0)} + \ldots)$; $ER^{(0)}$: $\boxed{\dot{v}^{(0)} = g, v^{(0)}(0) = 0} \Rightarrow v^{(0)}(t) = gt$

$ER^{(1)}$: $\boxed{\dot{v}^{(1)} = -g^2\alpha t\ ,\ v^{(1)}(0) = 0} \Rightarrow v^{(1)} = -\frac{1}{2}g^2\alpha t^2 \Rightarrow v(t) = gt - \frac{1}{2}g^2\alpha t^2$

(b) $(1 + \alpha v)\dot{v} = (v + \frac{\alpha}{2}v^2)^{\bullet} = (gt)^{\bullet} \Rightarrow v + \frac{\alpha}{2}v^2 = gt + C \stackrel{v(0) \stackrel{!}{=} 0}{=} gt$, $v^2 + \frac{2}{\alpha}v - \frac{2gt}{\alpha} = 0$

$\Rightarrow v(t) = -\frac{1}{\alpha} + \frac{1}{\alpha}\sqrt{1 + 2\alpha g t} = \frac{1}{\alpha}[-1 + 1 + \frac{1}{2}2\alpha g t - \frac{1}{8}(2\alpha g t)^2 + \ldots] \equiv$ (a) ✓