Klausur zu EMTP I (26. Mar. 2007)

Lösung

① gleichseitiges Dreieck hat ihn :  , $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$; $\sin\left(\frac{\pi}{3}\right) = \sqrt{1-c^2} = \frac{\sqrt{3}}{2}$

② $s \equiv \sin(\omega t),\ c \equiv \cos(\omega t)$; $\vec{r}(t) = R(1+s, c)$; $\dot{\vec{r}}(t) = R\omega\,(c, -s)$; $\ddot{\vec{r}}(t) = R\omega^2(-s,-c)$

③ $3m\ddot{z} = -2mg + mg$, $\boxed{\ddot{z} = -\frac{1}{3}g,\ \dot{z}(0)=0,\ z(0)=h}$

Ansatz $z = A + Bt^2 + \dots$, $z = h - \frac{1}{6}gt^2$; $z(t_1) \stackrel{!}{=} 0 \Rightarrow t_1 = \sqrt{6h/g}$

④ $E = T + V = \frac{m}{2}v^2 + \frac{k}{2}z^2 = \frac{m}{2}v^2 + \frac{k}{2}(b^2 - x^2)$

E-Erhaltung $E(x=0) \stackrel{!}{=} E(x=b)$, $\frac{k}{2}b^2 = \frac{m}{2}v^2 \Rightarrow v = b\sqrt{k/m}$

⑤ (a) Ans $v(t) = 1/\eta(t)$, $-\frac{\dot\eta}{\eta^2} = -\lambda\frac{1}{\eta^2}$, $\boxed{\dot\eta = \lambda,\ \eta(0) = \frac{1}{v_0}}$

Ans $\eta = A + Bt + \dots \Rightarrow \eta(t) = \frac{1}{v_0} + \lambda t$, $v(t) = \frac{1}{1/v_0 + \lambda t}$

(b) Ans $v(t) = [\eta(t)]^a$, $a\eta^{a-1}\dot\eta = -\lambda\eta^{na}$, $a = \frac{1}{1-n}$

also $v(t) = [\eta(t)]^{\frac{1}{1-n}}$, $\boxed{\dot\eta = -\lambda(1-n),\ \eta(0) = v_0^{1-n}}$

Ans $\eta = A + Bt + \dots \Rightarrow \eta(t) = v_0^{1-n} - \lambda(1-n)t$, $v(t) = [v_0^{1-n} - \lambda(1-n)t]^{\frac{1}{1-n}}$, $n=2 \Rightarrow$ (a) ✓✓

⑥ (a) $v = v_0 - \alpha t$, $0 \stackrel{!}{=} v_0 - \alpha t_1 \Rightarrow t_1 = v_0/\alpha$

(b) $(v\dot v) = \left(\frac{1}{2}v^2\right)^\cdot = -\beta = (\text{const}_t - \beta t)^\cdot \Rightarrow v = \sqrt{2(\text{const}_t - \beta t)} = \sqrt{v_0^2 - 2\beta t}$

$v(t=t_1) \stackrel{!}{=} 0 = \sqrt{v_0^2 - 2\beta t_1} \Rightarrow t_1 = v_0^2/2\beta$

⑦ $V = \frac{k_1}{2}x^2 + \frac{k_3}{2}z^2 + mgz$, $\vec{K} = (-\partial_x V, -\partial_z V) = (-k_1 x, -k_3 z - mg)$

⑧ (a) $H = H^T$ ✓; $(5-\lambda)(2-\lambda) - 4 = \lambda^2 - 7\lambda + 6 \stackrel{!}{=} 0$, $\lambda = \frac{7}{2} \pm \sqrt{49/4 - 6} \Rightarrow \lambda_1 = 1,\ \lambda_2 = 6$;

$Sp(H) = 7 = \lambda_1 + \lambda_2$ ✓; $\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{f}_1 = \frac{1}{\sqrt5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{f}_2 = \frac{1}{\sqrt5}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$;

$H' = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$, $D = \frac{1}{\sqrt5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, $\det D = 1$ ✓

(b) $\vec{c}\,' = D\vec{c} = c\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\ddot{\vec{r}}\,' = -\omega^2 H'\vec{r}\,' + \omega^2 c\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\omega^2\begin{pmatrix} x'-c \\ 6y' \end{pmatrix}$

⑨ (a) $\frac{1+\varepsilon}{1-\varepsilon^2} = \frac{1}{1-\varepsilon} = 1 + \varepsilon + \varepsilon^2 + \varepsilon^3 + O(\varepsilon^4)$ (geom. R.)

(b) $\ln\left(\frac{1}{\cos(\varepsilon)}\right) = -\ln(\cos(\varepsilon)) = -\ln\left(1 - \frac{\varepsilon^2}{2} + \frac{\varepsilon^4}{4!} - \dots\right) = \left(\frac{\varepsilon^2}{2} - \frac{\varepsilon^4}{4!} + \right) + \frac{1}{2}\left(\frac{\varepsilon^2}{2} - \dots\right)^2 + \dots = \frac{\varepsilon^2}{2} + \frac{\varepsilon^4}{12} + O(\varepsilon^6)$

(c) $\sinh\left(\sqrt{\varepsilon\sin(\varepsilon)}\right) = \sinh\left(\varepsilon\sqrt{1 - \varepsilon^2/6 + \dots}\right) = \sinh\left(\varepsilon - \frac{\varepsilon^3}{12} + \dots\right) = \left(\varepsilon - \frac{\varepsilon^3}{12} + \dots\right) + \frac{1}{6}(\varepsilon - \dots)^3 = \varepsilon + \frac{\varepsilon^3}{12} + O(\varepsilon^5)$

⑩ (a) $\boxed{\dot N^{(0)} = \alpha N^{(0)},\ N^{(0)}(0) = N_0} \Rightarrow N^{(0)} = N_0 e^{\alpha t}$

$\boxed{\dot N^{(1)} = \alpha N^{(1)} - \beta N_0^2 e^{2\alpha t},\ N^{(1)}(0) = 0}$, Ans $N^{(1)} = e^{\alpha t}u(t) \Rightarrow \boxed{\dot u = -\beta N_0^2 e^{\alpha t},\ u(0) = 0}$

$\Rightarrow u = \text{const}_t - \frac{\beta}{\alpha}N_0^2 e^{\alpha t}$, $\text{const}_t = \frac{\beta}{\alpha}N_0^2 \Rightarrow N^{(1)} = \frac{\beta}{\alpha}N_0^2 e^{\alpha t}(1 - e^{\alpha t})$

(b) $N(t) = \frac{N_0 e^{\alpha t}}{1 + \frac{\beta}{\alpha}N_0(e^{\alpha t} - 1)} \approx \underbrace{N_0 e^{\alpha t}}_{= N^{(0)} ✓✓} \underbrace{- N_0 e^{\alpha t}\frac{\beta}{\alpha}N_0(e^{\alpha t} - 1)}_{= N^{(1)} ✓✓} + O(\beta^2)$

Ergebnisse: ab 30. März ; online + Aushang (E6)