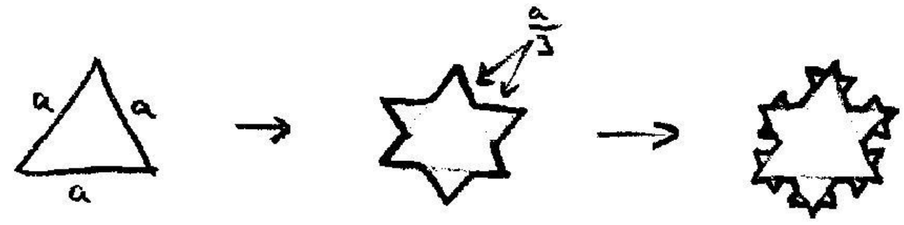
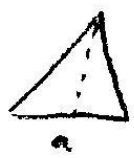
Potenzreihen : Koch'sche Schneeflocke

Konstruktionsvorschrift



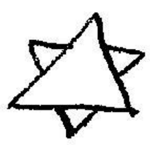
etc.

Frage: Flächeninhalt = ?

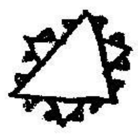


$$F(a) = \frac{a}{2}\sqrt{a^2-\frac{a^2}{4}} = \frac{a^2}{4}\sqrt{4-1} = a^2\,\frac{\sqrt{3}}{4}$$

$$F_0 = F(a)$$



$$F_1 = F_0 + 3\cdot F\left(\frac{a}{3}\right)$$



$$F_2 = F_1 + 3\cdot 4\cdot F\left(\frac{a}{3\cdot 3}\right)$$

$$F_3 = F_2 + 3\cdot 4\cdot 4\cdot F\left(\frac{a}{3\cdot 3\cdot 3}\right)$$

$$\vdots$$

$$F_n = F_{n-1} + 3\cdot 4^{n-1}\cdot F\left(\frac{a}{3^n}\right)$$

$$\vdots$$

$$F_\infty = F_0 + \sum_{n=1}^{\infty} 3\cdot 4^{n-1}\cdot F\left(\frac{a}{3^n}\right)$$

$$= a^2\frac{\sqrt{3}}{4} + \sum_{n=1}^{\infty} 3\cdot 4^{n-1}\cdot\frac{a^2}{3^{2n}}\,\frac{\sqrt{3}}{4}$$

$$= a^2\frac{\sqrt{3}}{4}\left(1+\frac{3}{4}\underbrace{\sum_{n=1}^{\infty}\left(\frac{4}{9}\right)^n}\right)$$

$$= \sum_{n=0}^{\infty}\left(\frac{4}{9}\right)^n - 1$$

geom. Reihe! (5.42) $\left(\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\right.$

$$= \frac{1}{1-\frac{4}{9}} - 1 = \frac{4}{5}$$

$$= a^2\frac{\sqrt{3}}{4}\,\frac{8}{5} = a^2\,\frac{2\sqrt{3}}{5}$$

Frage: Umfang = ?

$$U_0 = 3a$$

$$U_1 = \frac{4}{3}U_0$$

$$U_2 = \frac{4}{3}U_1 = \left(\frac{4}{3}\right)^2 U_0$$

$$\vdots$$

$$U_n = \left(\frac{4}{3}\right)^n U_0 \quad\Rightarrow\quad U_\infty = \infty\ !$$

Potenzreihen-Entwicklung, Ü32

Aufg.-Text: $m\ddot{x} = -\partial_x \Big(\overbrace{\kappa a^2 f\left(\tfrac{x}{a}\right)}^{=V(x)\ \text{"Potential"}}\Big)$

$\kappa a^2 f\left(\tfrac{x}{a}\right) \approx \text{const}_x + \frac{1}{2}\underline{\kappa_{eff}}\, x^2 + \dots$ was ist klein? $\frac{x}{a} \ll 1$, $x \ll a$

$m\ddot{x} = -\kappa_{eff}\, x + \dots$

Schw.-Glg: $\ddot{x} = -\omega^2 x$, Lsg $x(t) = A\sin(\omega t) + B\cos(\omega t)$

Vergleich m. oben: $\omega = \sqrt{\kappa_{eff}/m}$ (s. auch S. 45)

Vorbereitung: $f(s)$ gegeben; was ist klein? $s \ll 1$

$f(s) \approx c_0 + c_1 s + \underline{c_2} s^2 + \dots$

c_1 wahrscheinlich $= 0$, da alle f's gerade sind

$\Rightarrow \kappa_{eff} = 2\kappa c_2$

S. 43: $\cos(x) \approx 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$

$\sinh(x) \approx x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$ } z.B. leicht

$\cosh(x) \approx 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \dots$ } aus e-Reihe

$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

S. 44: $(1+x)^\lambda \approx 1 + \lambda x + \frac{\lambda(\lambda-1)}{2}x^2 + \dots$

(a) $f(s) = \frac{1}{s^2}\left(\cos(s) - \cos\left[\sinh(s)\right]\right)$

$\sinh(s) \approx s + \frac{s^3}{3!} + \dots$ ist klein!

$\cos[\sinh(s)] \approx 1 - \frac{1}{2}\left(s + \frac{s^3}{3!} + \dots\right)^2 + \frac{1}{4!}\left(s + \frac{s^3}{3!} + \dots\right)^4 + \dots$

$= 1 - \frac{1}{2}s^2\left(1 + \frac{s^2}{3!} + \dots\right)^2 + \frac{1}{4!}s^4\left(1 + \frac{s^2}{3!} + \dots\right)^4 + \dots$

$= 1 - \frac{1}{2}s^2\left(1 + 2\frac{s^2}{6} + \dots\right) + \frac{1}{24}s^4 + \dots$

$f(s) = \frac{1}{s^2}\left(\cancel{1} - \cancel{\frac{s^2}{2}} + \frac{s^4}{24} - \dots - \cancel{1} + \cancel{\frac{s^2}{2}} + \frac{s^4}{6} - \frac{s^4}{24} + \dots\right) = \underline{\underline{\frac{1}{6}}}\, s^2 + \dots$

(b) $f(s) = \frac{2\cosh(s)}{2 - \cosh^2(s)}$

$\approx \frac{2\left(1 + \frac{s^2}{2} + \dots\right)}{2 - \left(1 + \frac{s^2}{2} + \dots\right)^2} = \frac{2\left(1 + \frac{s^2}{2} + \dots\right)}{2 - \left(1 + s^2 + \dots\right)} = 2\left(1 + \frac{s^2}{2} + \dots\right)\left(1 - s^2 - \dots\right)^{-1}$

$= 2\left(1 + \frac{s^2}{2} + \dots\right)\left(1 + s^2 + \dots\right) = 2 + \underline{\underline{3}}\, s^2 + \dots$

(b)$_2$ $$f(s) = \partial_s \ln\left(\frac{1+\sinh(s)}{1-\sinh(s)}\right)$$

$\sinh(s) = s + \dots$ ist klein

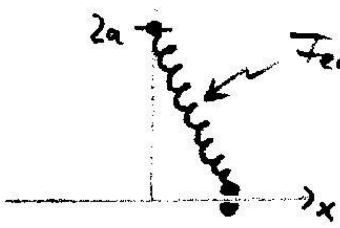
$$\begin{aligned}
\ln\left(\frac{\cdot\cdot}{\cdot\cdot}\right) &= \ln(1+\sinh(s)) - \ln(1-\sinh(s)) \qquad \text{(s. 5.41)}\\
&\approx \sinh(s) - \cancel{\frac{\sinh^2(s)}{2}} + \frac{\sinh^3(s)}{3} + \dots + \sinh(s) + \cancel{\frac{\sinh^2(s)}{2}} + \frac{\sinh^3(s)}{3} + \dots\\
&= 2\sinh(s) + \frac{2}{3}\sinh^3(s) + \dots\\
&\approx 2\left(s + \frac{s^3}{3!} + \dots\right) + \frac{2}{3}\left(s + \frac{s^3}{3!} + \dots\right)^3 + \dots\\
&= 2s + s^3 + \dots
\end{aligned}$$

$$\partial_s \ln\left(\frac{\cdot\cdot}{\cdot\cdot}\right) = 2 + \underline{\underline{3}}s + \dots \quad \checkmark \text{ s. } (b)_1$$

(c) $$f(s) = -\frac{\ln[1+\cosh(s)]}{\cosh(s)} \quad ; \quad \cosh(s) \approx 1 + \dots \ \text{nicht klein.}$$

$$\begin{aligned}
&\approx -\frac{\ln[1+1+\frac{s^2}{2}+\dots]}{1+\frac{s^2}{2}+\dots} = -\left(1+\frac{s^2}{2}+\dots\right)^{-1}\ln\left[2\left(1+\frac{s^2}{4}+\dots\right)\right]\\
&= -\left(1+\frac{s^2}{2}+\dots\right)^{-1}\left[\ln(2)+\ln\left(1+\frac{s^2}{4}+\dots\right)\right]\\
&\approx -\left(1-\frac{s^2}{2}-\dots\right)\left[\ln(2)+\left(\frac{s^2}{4}-\dots\right)\right]\\
&= -\ln(2) + s^2\underline{\underline{\left(\frac{1}{2}\ln(2)-\frac{1}{4}\right)}} + \dots
\end{aligned}$$

(d)



S.22 : $$V = \frac{\kappa}{2}\left(r_{m-Ref.} - l\right)^2 \overset{\text{hier}}{=} \frac{\kappa}{2}\left(\sqrt{4a^2+x^2} - a\right)^2 = \kappa a^2 \frac{1}{2}\left(\sqrt{4+\left(\frac{x}{a}\right)^2} - 1\right)^2$$

$$\begin{aligned}
\Rightarrow f(s) &= \frac{1}{2}\left(\sqrt{4+s^2} - 1\right)^2\\
&= \frac{1}{2}\left[2\left(1+\frac{s^2}{4}\right)^{\frac{1}{2}} - 1\right]^2\\
&\approx \frac{1}{2}\left[2\left(1+\frac{s^2}{8}+\dots\right)-1\right]^2 = \frac{1}{2}\left[1+\frac{s^2}{4}+\dots\right]^2 \approx \frac{1}{2}\left(1+\frac{s^2}{2}+\dots\right) = \frac{1}{2} + \underline{\underline{\frac{1}{4}}}s^2 + \dots
\end{aligned}$$

Störungsrechnung, Ü36 (a)

$$\boxed{\ddot{x} = 2a\omega^2 e^{-x/a} - \frac{1}{a}\dot{x}^2 \,, \quad \dot{x}(0)=0 \,, \quad x(0)=0}$$

hatten wir in Ü33 gelöst

per Ansatz $x(t) = a(c_2 t^2 + c_4 t^4 + c_6 t^6 + \dots)$ (Anf-Bed ✓✓)

$= a\left((\omega t)^2 - \frac{1}{2}(\omega t)^4 + \frac{1}{3}(\omega t)^6 \mp \dots\right)$ (*)

geraten $\overset{?}{=} a \ln(1+\omega^2 t^2)$

eingesetzt $\Rightarrow$ stimmt ✓

nun per Stö, ω^2 "klein"

$x(t) = x^{(0)}(t) + x^{(1)}(t) + \dots$

"genauso klein wie ω^2"

in ER: $\ddot{x}^{(0)} + \ddot{x}^{(1)} + \dots = 2a\omega^2 e^{-\frac{x^{(0)}+x^{(1)}+\dots}{a}} - \frac{1}{a}\left(\dot{x}^{(0)2} + 2\dot{x}^{(0)}\dot{x}^{(1)} + \dots\right)$

Koeff-Vergleich (0. Ordg)

$$\boxed{\ddot{x}^{(0)} = -\frac{1}{a}\dot{x}^{(0)2} \,, \quad \dot{x}^{(0)}(0)=0 \,, \quad x^{(0)}(0)=0}$$

$\left(\Rightarrow \dot{v} = -\frac{v^2}{a} \,,\ v(0)=0 \quad \Rightarrow v(t)=0\right) \Rightarrow x^{(0)}(t) = 0$

Koeff-Vergleich (1. Ordg)

$$\boxed{\ddot{x}^{(1)} = 2a\omega^2 \,, \quad \dot{x}^{(1)}(0)=0 \,, \quad x^{(1)}(0)=0}$$

$\Rightarrow x^{(1)}(t) = A + Bt + Ct^2 = a\omega^2 t^2$

vergleiche mit 1. Term in (*) ✓✓

Störungsrechnung, Ü36 (6)

$$\boxed{\dot v = -\lambda(1+\omega t)\,v\,, \quad v(0) = v_0}$$

exakte Lsg $\quad v(t) = v_0\, e^{-\lambda t - \frac{1}{2}\lambda\omega t^2}$

((check: $\quad \dot v = v_0\,(-\lambda - \lambda\omega t)\, e^{-\cdots} \;=\; -\lambda(1+\omega t)\,v$ ✓✓))

((oder: s.u., elegante Lsg per ln-Trick))

Q: worauf führt Stö. 1. Ordnung (in Gliedern ω)?

A: $\quad v(t) = \underline{v^{(0)}(t)} + v^{(1)}(t) \overset{?!}{=} v_0\, e^{-\lambda t}\,(\underline{1 - \tfrac{1}{2}\lambda\omega t^2})$

Q: nullte Ordg Stö durchführen?

A: $\quad \boxed{\dot v^{(0)} = -\lambda\, v^{(0)}\,, \quad v^{(0)}(0) = v_0}$

$$\Rightarrow\quad v^{(0)}(t) = v_0\, e^{-\lambda t}$$

Q: erste Ordnung Stö durchführen?

A: $\quad \dot v^{(1)} \overset{!}{=} -\lambda v^{(1)} - \lambda\omega t\, v^{(0)}$

$$\boxed{\dot v^{(1)} \overset{!}{=} -\lambda v^{(1)} - \lambda\omega t\, v_0 e^{-\lambda t}\,, \quad v^{(1)}(0) = 0}$$

wilde Dgl!

Substitution $\quad v^{(1)}(t) = e^{-\lambda t} u(t) \quad$ mit neuer Fkt u hilft ?!

in ER: $\quad \cancel{-\lambda e^{-\lambda t}} u + e^{-\lambda t}\dot u = \cancel{-\lambda e^{-\lambda t}} u - \lambda\omega t\, v_0 e^{-\lambda t}$

$\Rightarrow$ u-ER $\quad \boxed{\dot u = -\lambda\omega v_0 t\,, \quad u(0) = 0}$

$$\Rightarrow\quad u(t) = A + Bt + Ct^2 = -\tfrac{\lambda}{2}\omega v_0 t^2$$

also: $\quad v^{(1)}(t) = -\,e^{-\lambda t}\,\tfrac{\lambda}{2}\omega v_0 t^2$

Q: Vergleich mit Erwartung?

A: $\quad v^{(0)}(t) + v^{(1)}(t) = v_0\, e^{-\lambda t}\left(1 - \tfrac{\lambda}{2}\omega t^2\right)$ ✓✓ toll!