Introduction to Astronomy
Exercises week 15

31 January 2020

1. One definition of the habitable zone, is that region of space around a star, where the average temperature of a planetary body would be between 0°C and 100°C.

   (a) Calculate the habitable zone for the Solar System. To do so, equate the incoming flux from the Sun with the outgoing flux from the planet's black-body radiation (use the Stefan-Boltzmann law for this). In considering the incident solar radiation, assume an albedo (i.e. the fraction of incident light that is directly reflected) of 0.31, as for the Earth.

Solution:
The incident radiation from the Sun can be calculated from its luminosity (which in turn could have been derived from its black-body flux as given by the Stefan-Boltzmann law; integrated over its surface):

$$F_{\text{in}} = \frac{L_{\odot}}{4\pi D^2}$$

with D the distance between the Sun and the Earth (i.e. 1 AU). The energy absorbed by the Earth, then becomes:

$$E_{\text{in}} = F_{\text{in}}\pi R_{\text{Earth}}^2(1-A) = \frac{L_{\odot}R_{\text{Earth}}^2(1-A)}{4D^2}$$

where $A = 0.31$ is the Earth's albedo, i.e. the fraction of the incoming flux that is *not* absorbed. The energy lost in black-body radiation, is given by:

$$E_{\text{out}} = 4\pi R_{\text{Earth}}^2\sigma T_{\text{Earth}}^4$$

with $\sigma T_{\text{Earth}}^4$ the blackbody flux as defined by the Stefan-Boltzmann law.
Equating these two, we get:

$$4\pi R_{\text{Earth}}^2\sigma T_{\text{Earth}}^4 = \frac{L_{\odot}R_{\text{Earth}}^2(1-A)}{4D^2}$$

or, after re-writing:

$$D = \sqrt{\frac{L_{\odot}(1-A)}{16\pi\sigma T_{\text{Earth}}^4}} = \sqrt{\frac{3.846\times10^{26}\,\text{W}\times0.69}{16\pi 5.67\times10^{-8}\,\text{W/m}^2/\text{K}^4\times T_{\text{Earth}}^4}} = \frac{0.96\times10^{16}\,\text{mK}^2}{T^2}$$

For $T = 273\,\text{K}$, this becomes:

$$D_{\max} = 1.29\times10^{11}\,\text{m} = 0.86\,\text{AU}$$

and for $T = 373\,\text{K}$, we get:

$$D_{\min} = 6.94\times10^{10}\,\text{m} = 0.46\,\text{AU}.$$

This is obviously a suspicious result, because the Earth (which must be in the habitable zone), is not included. The reason this is the case, is because we ignored many details, the most important one being the greenhouse effect, which heats planets and therefore moves the habitable zone further out.

   (b) Same question, but now for an Earth-like planet orbiting Aldebaran, which has a temperature of 3910 K and a radius of 55.2 $R_{\odot}$.

Solution:
The total luminosity can be calculated from Stefan-Boltzmann's law, integrated over the surface of the star:

$$L_{\mathrm{A}} = \sigma T_{\mathrm{A}}^4 4\pi R_{\mathrm{A}}^2$$

where the subscript $_A$ indicates "Aldebaran". Hence, we can re-write our equation from question a) as follows:

$$D = \sqrt{\frac{\sigma T_{\mathrm{A}}^4 4\pi R_{\mathrm{A}}^2 (1-A)}{16\pi\sigma T_{\mathrm{planet}}^4}} = \sqrt{\frac{T_{\mathrm{A}}^4 R_{\mathrm{A}}^2 (1-A)}{4T_{\mathrm{planet}}^4}}.$$

Or:

$$D = \sqrt{\frac{3910^4 \left(55.2 \times 6.96 \times 10^8\right)^2 (1-A)}{4T_{\mathrm{planet}}^4}}.$$

From this, it follows that a first estimate of the habitable zone around Aldebaran would be between 11.7 AU and 21.8 AU. Note that this is again an underestimate, since the greenhouse effect would push this range to larger radii.

2. If there are N stars in a volume V, a fraction q of which are habitable, estimate the average distance between two habitable planets. (Hint: this is a trivial question: base it on the average volume per star.) Apply your solution to the Solar neighbourhood, where we have roughly 47 star (systems) within $520\,\mathrm{pc}^3$. Assume 1% and respectively 0.001% of these stars have habitable planets.

Solution:
There are two essentially equivalent approaches to this question. A relatively simple approach, is as follows:

The average volume per star, is V/N. To translate this volume in a typical distance, we can take the cubic root, giving us a distance of $\sqrt[3]{V/N}$. If only a fraction q of the planets is habitable, we include that factor as follows:

$$D \approx \sqrt[3]{\frac{V}{Nq}}.$$

Using $N = 47$, $V = 520\,\mathrm{pc}^3$ and $q = 0.01$, this gives:

$$D \approx 10.3\,\mathrm{pc}.$$

for 0.001% of star systems, this increases to 103 pc.

Alternatively, you could get a potentially more precise solution like this:

The stellar density is $\rho = Nq/V$. Estimating the radius of a sphere that would have exactly one star, given this density, we get:

$$N(R) = \rho\frac{4}{3}\pi R^3 = 1,$$

or:

$$R = \sqrt[3]{\frac{3}{4\pi\rho}} = \sqrt[3]{\frac{3V}{4\pi Nq}}.$$

From this, it follows that, for $q = 0.01$: $D = 6.4\,\mathrm{pc}$ and for $q = 0.001\%$, $D = 64\,\mathrm{pc}$.

Both approaches are acceptable since in reality the stellar distribution will be so inhomogeneous that whatever calculation we can make, will only ever get us an order-of-magnitude estimate at best. From that perspective, 103 pc is effectively identical to 64 pc.

3. (Off-topic question:) As you know, flux (expressed in W/m^2) is luminosity (in W) per area. In radio

astronomy, typically the unit of flux *density* is used instead, adding in the bandwidth of the radiation. In SI units, flux density would be expressed in $W/m^2/Hz$, but for any practical purpose, the *Jansky* is used, defined as: $1\,\mathrm{Jy} = 10^{-26}\,\mathrm{W/m^2/Hz}$, because most typical radio astronomy sources have fluxes of the order 1 Jy to 1 mJy. Now consider a microwave oven with a power of 1 kW, placed on the moon. Suppose its power is emitted across a bandwidth of 100 MHz, centred at 2.7 GHz. Calculate the flux density of this microwave oven, on Earth. Would this object be visible with a radio telescope?

Solution:

As described above, the flux density is defined as:

$$B_\nu = \frac{L}{4\pi D^2 \Delta\nu},$$

with B_ν the flux density, P the luminosity (or emitted power), D the distance and $\Delta\nu$ the bandwidth of the emission.

Using the distance between the Earth and the Moon, this becomes:

$$B_\nu = \frac{10^3\,\mathrm{W}}{4\pi\,(3.84 \times 10^8\,\mathrm{m})^2\,10^8\,\mathrm{Hz}} = 5.4 \times 10^{-24}\,\mathrm{W/m^2/Hz} = 540\,\mathrm{Jy}.$$

Given that this object is hundreds to thousands of times brighter than typical radio sources, it would easily be visible. In fact, it would outshine nearly everything else in the Universe!