Introduction to Astronomy
Exercises week 14

24 January 2020

1. In order to get a feeling for Olber's paradox, consider you're in a forest and try to evaluate how much of your view is blocked by trees. To do so, consider what fraction of directions (i.e. of the 360 degrees you can look around yourself) are blocked by trees, as a function of distance. (Clearly nearby you'll only see few trees and the fractional obstruction of your view is limited; considering longer distances, this blockage will accumulate through the combined effect of near and far trees.) Consider the number density of trees (measured in number of trees per square metre) to be N and the diameter of these trees to be D (typically). Express the fraction of directions blocked as $s(r)$ where r is the distance considered. Use your result to calculate the percentage of the horizon that is covered if there is on average 1 tree per square metre, with a typical diameter of 20 cm and you're in the middle of a forest with a radius of 20 m.

Solution:

If we consider a ring with thickness $\mathrm{d}r$ at average distance r, then the surface area of that ring is $2\pi r \mathrm{d}r$. Given that the density of trees is N, there will be $2\pi r \mathrm{d}r N$ trees in that ring. Now, if $s(r)$ is the fraction of this ring covered by trees closer to us than r, the total unobstructed length seen from this ring, is: $2\pi r\,(1 - s(r))$. Since each tree has a diameter D, the obstructed part of that visible stretch, is: $2\pi r \mathrm{d}r\,(1 - s(r))\,ND$.

So of the total $2\pi r$ circumference of this ring, a fraction $s(r)$ was already covered and a fraction

$$2\pi r \mathrm{d}r\,(1 - s(r))\,ND/\,(2\pi r) = \mathrm{d}r\,(1 - s(r))\,ND$$

is now added to this. Hence, over a distance $\mathrm{d}r$, the fraction covered increases by:

$$\mathrm{d}s = ND\,(1 - s(r))\,\mathrm{d}r.$$

Integrating this equation from out standpoint ($s = 0$, $r = 0$) to a distance r with $s(r)$, we get:

$$\begin{aligned} \int_0^s \frac{\mathrm{d}s(r)}{1 - s(r)} &= \int_0^r ND\mathrm{d}r \\ \ln\left(\frac{1}{1 - s(r)}\right) &= NDr \\ s(r) &= 1 - \mathrm{e}^{-NDr}. \end{aligned}$$

With the example of $N = 1\,\mathrm{m}^{-2}$, $D = 20 \times 10^{-2}\,\mathrm{m}$ and $r = 20\,\mathrm{m}$, we get: $s(r) = 98\%$ – which means you can hardly see any light through the trees! (This is why you should always stick to the paths!)

The relation to Olber's paradox is clear: stars may be less densely spread than trees, but the Universe (and the Galaxy) is a lot larger than your typical forest and because of our limited resolution, some stars can appear quite large. In fact, this effect is the reason why the Milky Way looks like a stream rather than a collection of individual stars, to the human eye. The fact that this doesn't cover the entire sky, is simply an illustration that the Galaxy is a disk, that the Universe has a finite size; and that distant Galaxies are few and faint.

2. Hubble's law can be used (in a somewhat simplistic way) to calculate distances based on radial velocities. Interpreting the redshift as $z = v_{\mathrm{rad}}/c$, calculate the redshift, distance, diameter and absolute magnitude of

a galaxy that has a radial velocity of 6940 km/s, an apparent magnitude of 14.4 and an angular diameter of 1.3 arcminutes.

Solution:

The redshift follows trivially from $z = v_{\rm rad}/c = 0.023$. Clearly this is not a very high redshift, so peculiar velocities of the galaxy may have a significant influence on the radial velocity. This would corrupt any of the following results.

Re-writing Hubble's formula, we have: $D = v/H$. With $H = 68\,\mathrm{km/s/Mpc}$ and $v = v_{\rm rad} = 6940\,\mathrm{km/s}$, this gives: $D = 102\,\mathrm{Mpc}$.

Using simple trigonometry, we can now derive the diameter:

$$\psi = \frac{1.3\pi}{60 \times 180} 102\,\mathrm{Mpc} = 39\,\mathrm{kpc}.$$

Finally, the absolute magnitude follows from its definition:

$$M = m - 5\log\frac{r}{10\,\mathrm{pc}} = 14.4 - 5\log 102 \times 10^5 = -20.6.$$

3. In the radiation-dominated early Universe (which we can approximate with a flat geometry), the temperature of the Universe evolved with time as follows:

$$T(t) = \left(\frac{3c^2}{32\pi G a}\right)^{1/4} t^{-1/2} = t^{-1/2} \times 1.52 \times 10^{10}\,\mathrm{Ks}^{1/2},$$

with G Newton's gravitational constant and a is the radiation constant ($a = 4\sigma/c$). Also, the Robertson-Walker scale factor R evolved as

$$R(t) = \left(\frac{32\pi G a T_0^4}{3c^2}\right)^{1/4} t^{1/2} = t^{1/2} \times 1.80 \times 10^{-10}\,\mathrm{s}^{-1/2},$$

with T_0 the temperature of the CMB. If the scale factor relates to redshift accoding to $1+z = R(t)^{-1}$, then calculate the redshift at which He nuclei were formed (this happens at a temperature of 10^9 K).

Solution:

Re-writing the temperature equation, we can get the time at which He nuclei formed, as:

$$t = \left(\frac{1.52 \times 10^{10}\,\mathrm{Ks}^{1/2}}{T}\right)^2 = 231\,\mathrm{s}.$$

This can provide us with the scale factor at that time:

$$R(t) = (231\,\mathrm{s})^{1/2} \times 1.8 \times 10^{-10}\,\mathrm{s}^{-1/2} = 2.7 \times 10^{-9}.$$

And, finally, leads to the redshift:

$$z = R^{-1} - 1 = 3.7 \times 10^8.$$