Introduction to Astronomy
Exercises week 11

20 December 2019

1. Two clusters are close to each other on the sky and are therefore assumed to have equal (or very similar) interstellar extinction properties. They have angular diameters of α and 3α respectively; and their distance moduli (i.e. apparent magnitude minus absolute magnitude) are 16.0 and 11.0, respectively. Assuming their diameters are equal and the interstellar extinction coefficient a is identical for both clusters, determine their distances and the value for a.

Solution:

Using the definition of absolute magnitude using interstellar extinction, we have a set of two equations:

$$m - M = 16 = 5\log\frac{D_1}{10\,\mathrm{pc}} + aD_1$$

and similar for D_2 with $m - M = 11$. Assuming the physical diameters are equal and knowing that their angular diameters differ by a factor of three, we get (furthermore using the small-angle approximation that $\sin\alpha = \alpha = diam/Dist$):

$$D_2 = \frac{D_1}{3}.$$

Hence, we have:

$$\begin{aligned} 16 &= 5\log\frac{D_1}{10\,\mathrm{pc}} + aD_1 \\ 11 &= 5\log\frac{D_1}{30\,\mathrm{pc}} + \frac{aD_1}{3}. \end{aligned}$$

subtracting three times the second equation from the first equation:

$$\begin{aligned} 16 - 33 &= 5\log\frac{D_1}{10\,\mathrm{pc}} + aD_1 - 15\log\frac{D_1}{10\,\mathrm{pc}} + 15\log 3 - aD_1 \\ -17 &= -10\log\frac{D_1}{10\,\mathrm{pc}} + 15\log 3. \end{aligned}$$

We can now solve for D_1 and find:

$$\begin{aligned} \frac{17}{10} + \frac{15}{10}\log 3 &= \log\frac{D_1}{10\,\mathrm{pc}} \\ 10^{17/10}10^{\log 3^{15/10}} &= \log\frac{D_1}{10\,\mathrm{pc}} \\ 10^{17/10}3^{15/10}10\,\mathrm{pc} &= D_1 \\ D_1 &= 2.6\,\mathrm{kpc}. \end{aligned}$$

Hence, $D_2 = D_1/3 = 0.87\,\mathrm{kpc}$.

Now filling this into one of the equation above, we can solve for a:

$$a = \frac{16 - 5\log 260.4}{2604\,\mathrm{pc}} = 1.5\,\mathrm{mag/kpc}.$$

2. Synchrotron radiation is produced when relativistic charged particles are deflected by interstellar magnetic fields. As an example, we consider a proton with a kinetic energy of $E_{\mathrm{kin}} = 1.6 \times 10^{-13}$ J moving perpendicular to a Galactic magnetic field with strength $B = 0.1\,\mathrm{nT}$. (Remember 1 Tesla = 1 Ws/Am^2). Given that the magnetic field exerts a force of $\vec{F} = q\vec{v} \times \vec{B}$ and that the centripetal force is mv^2/r, calculate the radius of the circular motion of the proton.

Solution:
From the equation for the centripetal force, we can derive the radius of the orbit:

$$r = \frac{mv^2}{qvB} = \frac{mv}{qB}$$

where we only consider the amplitudes of these parameters, since the magnetic field and the velocity vectors are perpendicular, so $\vec{v} \times \vec{B} = |v||B|$. The velocity we derive from the kinetic energy as follows:

$$v = \sqrt{\frac{2E_{\mathrm{kin}}}{m}} = \sqrt{\frac{3.2 \times 10^{-13}\,\mathrm{J}}{1.67 \times 10^{-27}\,\mathrm{kg}}} = 1.4 \times 10^7\,\mathrm{m/s}.$$

Now we can calculate the radius from the equation above:

$$r = \frac{1.67 \times 10^{-27}\,\mathrm{kg}\ 1.4 \times 10^7\,\mathrm{m/s}}{1.6 \times 10^{-19}\,\mathrm{C}\ 0.1 \times 10^{-9}\,\mathrm{T}} = 10^9\,\mathrm{m} = 0.01\,\mathrm{AU}.$$

3. There is a sharp decrease in the flux density of Cassiopeia A at a frequency of about 10 MHz. If this source is 3 kpc from the Sun and the average electron density is $0.03\,\mathrm{cm}^{-3}$, calculate whether the fall-off can be caused by the free-free absorption of free electrons along the line of sight. Assume the interstellar electron density is constant. (Hint: use the formulae for the emission measure and the optical depth of free-free emission to estimate the temperature the cloud should have in order to have an optical depth larger than unity. Compare this temperature to the typical temperature of ionised interstellar gas (thousands of Kelvins and up).)

Solution:
Assuming constant electron density, the emission measure is:

$$\mathrm{EM} = n_{\mathrm{e}}^2 \int_0^d \mathrm{d}l = n_{\mathrm{e}}^2 D = \left(0.03\,\mathrm{cm}^{-3}\right)^2 3000\,\mathrm{pc} = 2.7\,\mathrm{pc/cm}^6$$

This means the optical depth at 10 MHz becomes:

$$\tau_\nu = 8.235 \times 10^{-2} T^{-1.35} \times \left(10^{-2}\,\mathrm{GHz}\right)^{-2.1} 2.7\,\mathrm{pc/cm}^6 = 3.5 \times 10^3 \times T^{-1.35}.$$

So in order to have $\tau > 1$, we need: $3.5 \times 10^3 \times T^{-1.35} > 1$ and thus:

$$T < \left(\frac{1}{3.5 \times 10^3}\right)^{-1/1.35} \mathrm{K} = 424\,\mathrm{K}.$$

This is well below the temperature of the ionised medium: gas clouds at these temperatures (and below) will not have enough energy to be (or remain) ionised. Consequently bremsstrahlung cannot be the cause of the downturn in the spectrum of Cassiopeia A.

A more likely cause for the spectral break is self-absorption of the emission in the Cassiopeia A supernova remnant, where the temperature is much higher, but the densities are very much larger, too.