Introduction to Astronomy
Exercises week 10

13 December 2019

1. A globular cluster contains a million stars, each with an absolute magnitude equal to that of the Sun. If the cluster is 10 kpc away, calculate the total apparent magnitude of the cluster.

> **Solution:**
> Since we know the absolute magnitude of each cluster member to equal the Solar absolute magnitude, we can calculate the apparent magnitude based on:
>
> $$M_\odot = m + 5 - 5\log d$$
>
> with d the distance in pc. this gives:
>
> $$m = 4.8 - 5 + 5\log 10^4 = 19.8.$$
>
> Now, combining the 10^6 stars of the cluster, we need to consider that we have 10^6 times the flux, which means $F_{\rm cluster} = 10^6 F_{\rm star}$. So:
>
> $$m_{\rm cluster} - m_{\rm star} = -2.5\log\frac{10^6 F_{\rm star}}{F_{\rm star}} = -15.$$
>
> Therefore:
>
> $$m_{\rm cluster} = m_{\rm star} - 15 = 19.8 - 15 = 4.8.$$

2. The Pleiades cluster contains 230 stars within a radius of 4 pc. Assume that all stars weigh one Solar mass and that they are all separated by 2 pc.

   (a) Calculate the average potential and kinetic energy for each star and use the virial theorem ($\langle E_{\rm kin}\rangle = -0.5\langle E_{\rm pot}\rangle$) to estimate the typical velocity of stars in this cluster.

> **Solution:** The kinetic energy for a single star can easily be described in terms of the average velocity:
>
> $$\langle E_{\rm kin}\rangle = \frac{1}{2}M_\odot \bar{v}^2$$
>
> with $\bar{v}$ the average velocity of all stars.
> The potential energy for a couple of stars, is $-GM_\odot M_\odot/R$ so adding this for all $N(N-1)/2$ couples, we get:
>
> $$E_{\rm pot}^{\rm tot} = -\frac{GM_\odot^2 N(N-1)}{2R}.$$
>
> Using the virial theorem, we get:
>
> $$\langle E_{\rm kin}\rangle = \frac{1}{2}M_\odot \bar{v}^2 N = \frac{GM_\odot^2 (N-1)N}{4R} = \frac{1}{2}\langle E_{\rm pot}\rangle$$
>
> therefore:
>
> $$\bar{v}^2 = \frac{GM_\odot(N-1)}{2R}$$

and

$$\bar{v} = \sqrt{\frac{GM_{\odot}(N-1)}{2R}} = \sqrt{\frac{6.67259\times10^{-11}\,\mathrm{Nm^2/kg^2}1.99\times10^{30}\,\mathrm{kg}229}{2\times2\times3.0857\times10^{16}\,\mathrm{m}}} = 496\,\mathrm{m/s}.$$

(b) Now calculate the escape velocity of this cluster, at a distance of 2 pc.

Solution:
The escape velocity is simply:

$$v_{\mathrm{esc}} = \sqrt{\frac{2GMN}{R}} = \sqrt{\frac{2\times6.67259\times10^{-11}\,\mathrm{Nm^2/kg^2}1.99\times10^{30}\,\mathrm{kg}230}{2\times3.0857\times10^{16}\,\mathrm{m}}} = 995\,\mathrm{m/s}.$$

Assuming equal division of kinetic energy, one could imagine that the least massive stars (which get the highest velocities) could escape the cluster, but clearly the more massive ones probably will remain in the cluster until they end their lives.

3. One of the (distance-insensitive) ways to determine the age of a globular cluster, is by measuring the offset between its turnoff point and its horizontal branch: $\Delta V = M_{\mathrm{V}}(\mathrm{TO}) - M_{\mathrm{V}}(\mathrm{HB})$. However, this measure is not only dependent on the age, but also on the metallicity of the cluster:

$$\Delta V = 2.7\log\left(\frac{t}{\mathrm{Gyr}}\right) + 0.13[\mathrm{Fe/H}] + 0.59.$$

Estimate by how much the metallicity [Fe/H] must be off in order to produce a 10% error in the age of the cluster. Note that the metallicity is defined as follows:

$$\left[\frac{\mathrm{Fe}}{\mathrm{H}}\right] = \log\left(\frac{N_{\mathrm{Fe}}}{N_{\mathrm{H}}}\right)^{\mathrm{Star}} - \log\left(\frac{N_{\mathrm{Fe}}}{N_{\mathrm{H}}}\right)^{\mathrm{Sun}}$$

where log is the logarithm base 10.

Solution: The measured ΔV value would remain constant, so we can write:

$$2.7\log(t_{\mathrm{true}}) + 0.13[\mathrm{Fe/H}]_{\mathrm{true}} + 0.59 = 2.7\log(t_{\mathrm{wrong}}) + 0.13[\mathrm{Fe/H}]_{\mathrm{wrong}} + 0.59$$

Now, since $t_{\mathrm{true}} = 0.9 t_{\mathrm{wrong}}$ (alternatively, 0.9 should be replaced by 1.1), we get:

$$2.7\log(1/0.9) = 0.13\log\left(\frac{(N_{\mathrm{Fe}}/N_{\mathrm{H}})^{\mathrm{wrong}}}{(N_{\mathrm{Fe}}/N_{\mathrm{H}})^{\mathrm{true}}}\right)$$

hence:

$$\left(\frac{N_{\mathrm{Fe}}}{N_{\mathrm{H}}}\right)^{\mathrm{wrong}} = 0.9^{-2.7/0.13}\left(\frac{N_{\mathrm{Fe}}}{N_{\mathrm{H}}}\right)^{\mathrm{true}} = 8.92\left(\frac{N_{\mathrm{Fe}}}{N_{\mathrm{H}}}\right)^{\mathrm{true}}.$$

Alternatively, if the age is underestimated (i.e. the 0.9 factor becomes 1.1), we get that the 8.92 factor becomes 0.14.

In conclusion, to get the cluster age overestimated by 10%, the metallicity has to be overestimated by a factor of nearly 9. To get the cluster age underestimated by 10%, the metallicity has to be underestimated by $\sim$ 85%. So even a very imprecise estimate of the cluster metallicity allows the cluster age to be estimated to much better than 10% precision.