Introduction to Astronomy
Exercises week 9

6 December 2019

1. The Crab pulsar (PSR B0531+21) has a spectral index of -3 over the frequency range from 10 MHz to 10 GHz (i.e. $S \propto \nu^{-3}$). If the measured flux at 400 MHz is 650 mJy and the spin-down luminosity $\mathrm{d}E/\mathrm{d}t$ is given by

$$\mathrm{d}E/\mathrm{d}t = 4\pi^2 I P^{-3} \mathrm{d}P/\mathrm{d}t$$

with moment of inertia $I = 10^{45}\,\mathrm{g\,cm^2}$, spin period $P = 33\,\mathrm{ms}$ and spindown $\mathrm{d}P/\mathrm{d}t = 4.23 \times 10^{-13}\mathrm{s/s}$, calculate what fraction of energy loss the radio emission (between 10 MHz and 10 GHz) accounts for. The Crab pulsar is at a distance of approximately 2 kpc. (Hint: calculate the energy loss derived from the spin-down luminosity. Also calculate the total power emitted in the radio by integrating the flux over frequency and over a sphere, assuming isotropic emission. Then compare these two values.)

Solution: We are asked to compare the total flux observed with that expected from spin-down. The total energy loss from spin-down is:

$$\dot{E} = 4\pi^2 I \dot{P} P^{-3} = 4\pi^2 10^{38}\mathrm{kg\,m^2} 4.23 \times 10^{-13} \left(33 \times 10^{-3}\mathrm{s}\right)^{-3} = 4.6 \times 10^{31}\,\mathrm{W}.$$

This we now have to compare to the total energy emitted in radio. To do this, we first need to derive the expression for the flux as a function of frequency. We know that this relation is a power-law with index -3 and that at 400 MHz, we have a flux of 650 mJy. Therefore:

$$S(\nu) = 0.650\,\mathrm{Jy} \left(\frac{\nu}{4 \times 10^8\,\mathrm{Hz}}\right)^{-3}$$

So this allows us to calculate the total energy emitted in the radio band:

$$L = 4\pi D^2 \int_{10\,\mathrm{MHz}}^{10\,\mathrm{GHz}} S(\nu)\mathrm{d}\nu.$$

With a distance of $D = 2\,\mathrm{kpc} = 6.17 \times 10^{19}\,\mathrm{m}$, we get:

$$L = \frac{4\pi \left(6.17 \times 10^{19}\mathrm{m}\right)^2 0.650\mathrm{Jy}}{\left(4 \times 10^8 \mathrm{Hz}\right)^{-3}} \int_{10\mathrm{MHz}}^{10\mathrm{GHz}} \nu^{-3}\mathrm{d}\nu = 1.99 \times 10^{66}\mathrm{m^2 Jy Hz^3} \left[\frac{-\nu^{-2}}{2}\right]_{10\mathrm{MHz}}^{10\mathrm{GHz}}$$

with $1\mathrm{Jy} = 10^{-26}\mathrm{W/m^2/Hz}$, we get:

$$L = 1.99 \times 10^{40}\mathrm{WHz^2} \left(\frac{10^{-14}\mathrm{Hz}^{-2}}{2} - \frac{10^{-20}\mathrm{Hz}^{-2}}{2}\right) = 1.0 \times 10^{26}\mathrm{W}.$$

So the radio emission accounts for $10^{26}/\left(4.6 \times 10^{31}\right) = 2 \times 10^{-4}\%$ of the total energy loss. This is a negligible amount, but that can be explained by the energy of the photons, which scales with their frequency. Given that pulsars also emit gamma rays (which have frequencies thirteen orders of magnitudes higher than radio waves), suggests most of the energy loss is carried by gamma ray photons. However, the number of photons emitted in radio is much higher, which is why pulsars tend to be detected in radio far more easily than in gamma rays (also because gamma rays cannot be observed from Earth).

2. A Cepheid variable has a period of 20 days and a mean apparent magnitude of 20 mag. What is its distance?

Solution: Given the periodicity and the period-magnitude relation for Cepheid variables, we can calculate the absolute magnitude:

$$\langle M \rangle = -2.78 \log \frac{P}{10\,\text{days}} - 4.13 = -2.78 \log 2 - 4.13 = -5.0.$$

Given the absolute magnitude and the apparent magnitude, we can now calculate the distance by re-writing the definition of the absolute magnitude:

$$m - M = 5 \log \frac{D}{10\,\text{pc}}$$

hence:

$$D = 10\,\text{pc}\left(10^{m-M}\right)^{0.2} = 10\,\text{pc}\left(10^{20+5}\right)^{0.2} = 10^6\,\text{pc}.$$

So the star is 1 Mpc away.

3. Cassiopeia A is a supernova remnant (SNR) with an angular diameter of $5.5'$ at a distance of 3 kpc.

   (a) If the observed expansion velocity is 6.8×10^6 m/s, calculate the expected age of the remnant, assuming a constant expansion velocity.

   Solution: The radial expansion of the nebular is:

   $$\frac{5.5\,\text{arcmin}}{2 \times 60 \times 180} \pi \times 3\,\text{kpc} = 2.4\,\text{pc} = 7.4 \times 10^{16}\,\text{m}.$$

   At an average velocity of 6.8×10^6 m/s, this would take:

   $$T = \frac{7.4 \times 10^{16}\,\text{m}}{6.8 \times 10^6\,\text{m/s}} = 1.09 \times 10^{10}\,\text{s} = 345\,\text{yr}.$$

   So the supernova probably occured sometime around 1669. (Note: since the star is surrounded by a dense cloud of gas and dust, the actual supernova was not observed at the time.)

   (b) If we have a resolution of 50 mas[1], how long would we need to observe in order to detect the expansion of the nebula?

   Solution:
   50 mas at 3 kpc distance corresponds to a linear size of

   $$\frac{50 \times \pi}{3600 \times 1000 \times 180} \times 3 \times 10^3\,\text{pc} = 7.3 \times 10^{-4}\,\text{pc} = 2.2 \times 10^{13}\,\text{m}.$$

   If the remnant is expanding at 6.8×10^6 m/s, then an expansion on this scale would take:

   $$\frac{2.2 \times 10^{13}\,\text{m}}{2 \times 6.8 \times 10^6\,\text{m/s}} = 1.6 \times 10^6\,\text{s} = 19\,\text{days}.$$

   So the expansion of the SNR should be visible on the time scale of a month!

4. A variable star changes its magnitude by 2 mag. If its effective temperature is 6000 K at the maximum and 5000 K at the minimum, how much does its radius change?

[1] mas = milli-arcsecond

Solution:

The magnitude is related to the flux and the flux is related to the fourth power of the temperature (Stefan-Boltzmann's law integrated over the stellar surface). Hence:

$$\Delta m = -2.5 \log F_1/F_{\text{ref}} + 2.5 \log F_2/F_{\text{ref}} = -2.5 \log \frac{4\pi R_1^2 \sigma T_1^4}{4\pi R_2^2 \sigma T_2^4} = -2.5 \log \frac{R_1^2 T_1^4}{R_2^2 T_2^4}$$

Therefore we get the:

$$10^{-\Delta m/2.5} = \frac{R_1^2 T_1^4}{R_2^2 T_2^4}$$

or:

$$\frac{T_2^4}{T_1^4} 10^{-\Delta m/2.5} = \frac{R_1^2}{R_2^2}.$$

Hence:

$$\frac{R_1}{R_2} = \frac{T_2^2}{T_1^2} 10^{-\Delta m/5} = \frac{6000^2}{5000^2} 10^{-0.4} = 0.57.$$

So the stellar radius decreases by 43%.

Note the difference in the application of Stefan-Boltzmann's law this week and last week. The reason we need the distance in our present calculation and did not need the radius of the star in last week's calculation, is due to an easy point of confusion related to the flux densities in consideration. Specifically, when we discuss the Stefan-Boltzmann law generally:

$$F_{\text{blackbody}} = \sigma T^4,$$

we consider the flux density *leaving* the body. That means that the total emitted power must be integrated over the size of the object:

$$P_{\text{total}} = 4\pi R^2 \sigma T^4,$$

where R is the radius of the star.

However, in this week's exercise, we discuss the apparent magnitude:

$$m = -2.5 \log \frac{F_{\text{obs}}}{F_{\text{ref}}},$$

where F_{obs} is the flux density *received* by us on Earth. Consequently, in terms of the total power emitted by the source, we have:

$$F_{\text{obs}} = \frac{P_{\text{total}}}{4\pi D^2},$$

with D the distance to the source.

Putting these things together, we get:

$$F_{\text{obs}} = \frac{4\pi R^2 \sigma T^4}{4\pi D^2} = \frac{R^2}{D^2} \sigma T^4.$$

Consequently, when referring the apparent magnitude to the temperature of the star, we strictly speaking need to consider both the radius and the distance of the star in question. In last week's exercise both the distance and the radius of the two stars we compared were identical, so this all cancelled out. In the exercise this week, the radius does change and hence needs to be taken into account.