Introduction to Astronomy
Exercises week 8

29 November 2019

1. The light curve (i.e. intensity – or magnitude – as a function of time) of an eclipsing binary is mostly flat, except at superior and inferior conjunction, when one of the stars moves in front of the other one, causing a complete eclipse. At these times only the brightness of the eclipsing star can be seen. Using apparent magnitudes, these values are m_A and m_B respectively; the combined apparent magnitude (away from the eclipses) is m_Tot. If the effective temperatures of the stars are $T_\mathrm{A} = 5000\,\mathrm{K}$ and $T_\mathrm{B} = 12000\,\mathrm{K}$ respectively, calculate the size of the eclipses in magnitudes (i.e. calculate $m_\mathrm{B} = m_\mathrm{Tot} - m_\mathrm{A}$ and $m_\mathrm{A} = m_\mathrm{Tot} - m_\mathrm{B}$). Assume the stars have equal radii.

Solution:
According to the definition, $m = -2.5 \log F/F_\mathrm{ref}$ and so $m_\mathrm{Tot} - m_\mathrm{A} = -2.5 \log F_\mathrm{Tot}/F_\mathrm{A}$.
The fluxes we can get from integrating Stefan-Boltzmann's law over the surfaces of the stars: $F_\mathrm{A} = \sigma T_\mathrm{A}^4$, $F_\mathrm{B} = \sigma T_\mathrm{B}^4$ and thus: $F_\mathrm{Tot} = \sigma \left(T_\mathrm{A}^4 + R_\mathrm{B}^4\right)$. Hence, we have:

$$m_\mathrm{B} = m_\mathrm{Tot} - m_\mathrm{A} = -2.5 \log \left(\frac{T_\mathrm{A}^4 + T_\mathrm{B}^4}{T_\mathrm{A}^4}\right) = -3.8.$$

Similarly, we get $m_\mathrm{A} = m_\mathrm{Tot} - m_\mathrm{B} = -0.03$.

2. A planet with mass m orbits a star with mass M at a distance of a in a circular orbit. If the distance of the Star's centre to the common centre of mass is a', then show that:

$$MP^2 = a^2(a - a'),$$

where P is the orbital period in years, distances are in AU and masses are in Solar masses.

Solution:
We have Kepler's third law:

$$P^2 M_\mathrm{Tot} = a_\mathrm{comb}^3$$

and from geometry, we know: $M/m = a_1/a'$ where a_1 is the semi-major axis of the planet's orbit. (Note that $a_1 \neq a_\mathrm{comb}$!). Also, $a_\mathrm{comb} = a_1 + a' = a$. So we get:

$$P^2 M \left(1 + \frac{a'}{a_1}\right) = a^3.$$

This can be rewritten as follows:

$$P^2 M \frac{a_1 + a'}{a_1} = a^3.$$

or:

$$MP^2 = a^2 a_1 = a^2(a - a').$$

3. (a) If the Sun were to collapse into a neutron star with a radius of 8 km, what would be its density? (Assume no mass loss.)

Solution: The density is given by mass per volume. The mass of the Sun is given, the volume would be $V = 4/3\pi R^3$ with $R = 8 \times 10^3$ m. Hence:

$$\rho = \frac{3M_\odot}{4\pi R^3} = \frac{3 \times 1.99 \times 10^{30}\,\text{kg}}{4\pi\left(8 \times 10^3\right)^3\,\text{m}^3} = 9 \times 10^{17}\,\text{kg/m}^3.$$

As a point of reference, the density of the Sun is currently around $1\,\text{g/cm}^3$.

(b) What would be its rotational period if we assumed conservation of angular momentum and assumed that both the Sun and the pulsar are homogeneous spheres? (Remember the rotational period of the Sun is about a month – you can assume 30 days.)

Solution:
Using conservation of angular momentum ($L = I\omega = const$) and assuming homogeneous spheres for both the Sun and the pulsar, we get:

$$R_\odot^2 \omega_\odot = R_{\text{PSR}}^2 \omega_{\text{PSR}}$$

so, with $\omega = 2\pi/P$:

$$P_{\text{PSR}} = \left(\frac{R_{\text{PSR}}}{R_\odot}\right)^2 P_{\text{Sun}}.$$

Hence:

$$P_{\text{PSR}} = \left(\frac{8\,\text{km}}{6.96 \times 10^5\,\text{km}}\right)^2 30\,\text{days} = 3.96 \times 10^{-9}\,\text{days} = 3 \times 10^{-4}\,\text{s}.$$

So the rotational period would be 0.3 ms. This is much smaller (by a factor of about 4) than any known pulsar (and about 100 times smaller than the spin period of young pulsars – the Crab pulsar has a spin period of ~33 ms). The reason probably lies in the inhomogeneous nature of the Sun; the likelihood of significant mass loss during the collapse; and the loss of angular momentum in the process of collapsing (e.g. through magnetic coupling with the dust cloud thrown off in the process of imploding).

4. The escape velocity of an object is given as $v_{\text{esc}} = \sqrt{2GM/R}$. If the Sun were to turn into a black hole (without mass loss), how small would it need to be?

Solution:
The Sun would be a black hole if its escape velocity equalled the speed of light. Ergo:

$$R \leqslant 2GM/c^2.$$

For the Sun, this becomes:

$$R \leqslant \frac{2 \times 6.67259 \times 10^{-11} \times 1.99 \times 10^{30}}{\left(3 \times 10^8\right)^2}\,\text{m} = 2951\,\text{m}.$$

So slightly less than 3 km radius.