Introduction to Astronomy
Exercises week 7

22 November 2019

1. An interstellar gas cloud (which we approximate as a homogeneous sphere) has a mass of $1\,M_\odot$ and density of 10^{10} atoms per cm^3. Its rotation period is 1000 years.

   (a) What is the rotation period after the cloud has condensed (in its entirety) into a star of solar size?

   Solution:
   Assuming conservation of angular momentum, and knowing that angular momentum is given by $L = I\omega$ where I is the moment of inertia and ω is the angular veolicty, we can compare the gas cloud and the Sun as follows:

   $$L = I_\odot \omega_\odot = I_{\mathrm{gas}} \omega_{\mathrm{gas}}.$$

   Now we use the equation for the moment of inertia (equation 1), assuming that both the Sun and the cloud are homogenous spheres; we furthermore convert angular velocity into rotational period ($\omega = 2\pi/P$) and, knowing that the cloud and the Sun have equal mass, we get:

   $$\frac{R_\odot^2}{P_\odot} = \frac{R_{\mathrm{gas}}^2}{P_{\mathrm{gas}}}.$$

   The rotational period of the cloud is given (1000 years) and the radius of the Sun is given in the appendix (formula sheet). So all we now need, is the radius of the cloud. Knowing that the cloud is spherical and weighs one Solar mass, we can get the radius of the cloud from its volume and density:

   $$M = \frac{4}{3}\pi R^3 \rho = M_\odot$$

   therefore:

   $$R_{\mathrm{gas}} = \left(\frac{3M_\odot}{4\pi\rho}\right)^{1/3} = \left(\frac{5.97 \times 10^{33}\,\mathrm{g}}{4\pi 1.67 \times 10^{-24}\,\mathrm{g} \times 10^{10} 10^{6}\,\mathrm{m}^{-3}}\right)^{1/3} = 3.05 \times 10^{13}\,\mathrm{m}.$$

   Therefore, the rotation period after collapse, becomes:

   $$P_\odot = P_{\mathrm{gas}} \frac{R_\odot^2}{R_{\mathrm{gas}}^2} = 1000\,\mathrm{yr} \left(\frac{6.96 \times 10^8}{3.05 \times 10^{13}}\right)^2 = 5.2 \times 10^{-7}\,\mathrm{yr} = 0.045\,\mathrm{hr} = 16\,\mathrm{s}.$$

   (b) In class we saw that the Sun's actual rotation is of the order of a month. The mismatch with the value from the previous question can be explained in two ways: either the assumed rotation period of the cloud is grossly incorrect, or the conservation of angular momentum during the collapse, does not work perfectly. Given that typical values for gas motions in the interstellar medium are tens of km/s – and noting that these are *random* velocities, i.e. that the variance of the velocity is of the same order, explain if the rotation of the gas cloud could be grossly overestimated; and whether this could explain the mismatch. (Hint: calculate the typical velocity such a rotation corresponds to and check if that is comparable to (i.e. of the same order as) the typical gas motions in the interstellar medium.)

Solution:
The rotation period P of 1000 yrs for a cloud with a radius of $R = 3 \times 10^{13}$ m corresponds to a velocity of $2\pi R/P = 2\pi 3 \times 10^{13} / (1000 \times 365.25 \times 24 \times 3600)$ km/s $= 5.97$ km/s. This is for the edge of the cloud, which is dominant in terms of angular momentum.This is somewhat smaller than the typical gas velocities of tens of km/s and so could be realistic.

Furthermore, the mismatch between the answer from the previous question and the actual rotational period of the Sun (~ 1 month $\approx 30 \times 24$ hr $= 720$ hr), is five orders of magnitude. Given that this rotational period scales linearly with the rotational period of the cloud, in order for conservation of angular momentum to work perfectly, the rotational period of the cloud would have to be of the order of 1000×10^5 yrs, i.e. about hundred million years, corresponding to gas velocities of $\sim 6 \times 10^{-5}$ km/s, i.e. six orders of magnitude smaller than the typical gas velocities. While the random motion does average out to some degree, it seems uncanny for it to cancel out so extremely well.
We conclude that the suggested rotational period of the cloud is realistic, perhaps a bit short, but can definitely not be the only explanation for the mismatch. Clearly the collapse of the protostellar cloud does not conserve angular momentum perfectly, as a lot of angular momentum must have been lost somehow. (Perhaps through magnetic breaking?)

Note: the final paragraph of this question is somewhat speculative and therefore not fully required. The crux of the question is to perform a sanity check on the hypothetical cloud in question 1a. Whether random motions of order tens of km/s give rise to an angular momentum comparable to that of the cloud, or to that of the Sun, is a statistical answer that requires some experience or a numerical simulation. The rest of the answer is needed, though, because it indicates what to look at when trying to figure out what explains the observed mismatch.

2. The nuclear time scale for a particular star, scales (only) with M/L; and the thermal time scale is proportional to $M^2/(RL)$ (only), where M is the mass of the star, L is its luminosity and R is its radius.
   (a) Given that for the Sun the nuclear time scale is 10 billion years (10^{10} yrs) and its thermal time scale is 20 million years, derive exact equations for both quantities. (Hint: choose your units carefully.)

Solution:
Expressing the proportionalities in Solar units, allows us to define the proportionality constant as the relevant time for the Sun:

$$t_{\text{nucl}} = 10^{10} \frac{M_{M_\odot}}{L_{L_\odot}} \text{ yr}$$

and

$$t_{\text{therm}} = 2 \times 10^7 \frac{M^2_{M_\odot}}{R_{R_\odot} L_{L_\odot}} \text{ yr.}$$

Alternative ways to write this, are:

$$\frac{t_{\text{nucl}}}{1\,\text{yr}} = 10^{10} \frac{M}{1\,M_\odot} \left(\frac{L}{L_\odot}\right)^{-1}.$$

and

$$\frac{t_{\text{therm}}}{1\,\text{yr}} = 2 \times 10^7 \left(\frac{M}{1\,M_\odot}\right)^2 \left(\frac{R}{1\,R_\odot} \frac{L}{1\,L_\odot}\right)^{-1}$$

Note: You could also use SI units and derive the proportionality constant. This is equally correct, but less practical since stellar parameters are typically (as in the following exercise) expressed in Solar units anyway.

   (b) Given for Vega its mass of $2\,M_\odot$, radius of $3\,R_\odot$ and luminosity of $60\,L_\odot$, calculate its nuclear and thermal timescales. If you know that Vega is a star of spectral class A0V, what do you expect for those timescales?

Solution:
The numeric answers follow straight from the solutions of the previous exercise:

$$t_{\text{nucl}} = 10^{10}\frac{2}{60}\,\text{yr} = 3.33 \times 10^{8}\,\text{yr}$$

and

$$t_{\text{therm}} = 2 \times 10^{7}\frac{2^2}{3 \times 60}\,\text{yr} = 4.4 \times 10^{5}\,\text{yr}.$$

Because Vega is a brighter, hotter, star (class A0), we expect it to burn its fuel more rapidly and so we expect its timescales to be shorter. The fact that it is in luminosity class V (dwarf stars) doesn't really tell us much about its lifetime, it mostly just indicates that it is still on the main sequence, which means it must be younger than 3 million years and therefore younger than (or at most of roughly the same age as) our Sun.