Introduction to Astronomy
Exercises week 6

15 November 2019

1. Flux density can be defined in two different ways: either it can be considered to be the power per unit collecting area and unit bandwidth, or as the power per unit collecting area and unit wavelength. Both definitions are common, depending on which area of astronomy you consider, so it is important to know how to convert from one definition to the other.

   If the blackbody spectrum in terms of bandwidth is given as

   $$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(h\nu/kT\right) - 1}$$

   then derive the blackbody spectrum in terms of wavelength. (Hint: consider $|B_\nu(T)\mathrm{d}\nu| = |B_\lambda(T)\mathrm{d}\lambda|$.)

   Solution:
   Because $|B_\nu(T)\mathrm{d}\nu| = |B_\lambda(T)\mathrm{d}\lambda|$, we have:

   $$B_\lambda(T) = B_\nu(T)\left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right|.$$

   We also know the relation between ν and λ: $\nu = c/\lambda$, so $\mathrm{d}\nu/\mathrm{d}\lambda = -c/\lambda^2$. Hence:

   $$B_\lambda(T) = cB_\nu(T)/\lambda^2 = c\frac{2h\nu^3}{c^2\lambda^2}\frac{1}{\exp\left(h\nu/kT\right) - 1} = \frac{2hc^2}{\lambda^5}\frac{1}{\exp\left(hc/k\lambda T\right) - 1}.$$

2. Last week, we saw how faint most stars are in the radio part of the spectrum (i.e. in the frequency range $\nu \lesssim 10^{11}$ Hz).

   (a) Given that the peak of the blackbody spectrum for stars typically falls in the optical range ($\nu \approx 10^{15}$ Hz), derive the Wien approximation to the Planck spectrum given in the previous exercise. (Hint: Wien's approximation simplifies Planck's blackbody spectrum by moving the exponential to the numerator [i.e. get rid of the division so that the exponential ends up being merely a multiplicative factor].)

   Solution:
   The Planck blackbody spectrum was given in exercise 1:

   $$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(h\nu/kT\right) - 1}.$$

   The exponent of the exponential is: $h\nu/kT$ with $h = 6.626 \times 10^{-34}$ Js Planck's constant and $k = 1.380658 \times 10^{-23}$ J/K Boltzmann's constant. Therefore: $h\nu/kT = 4.8 \times 10^{-11}\nu/T$. So for $\nu \gg 10^{11}$, we have: $h\nu/kT \gg 1$ (since the temperature is at most a few thousand degrees) and therefore: $e^{h\nu/kT} \gg 1$. We can then make the approximation:

   $$\exp\left(h\nu/kT\right) - 1 \approx \exp\left(h\nu/kT\right).$$

With this approximation, the blackbody law becomes:

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\exp\left(-h\nu/kT\right).$$

This is Wien's approximation.

(b) Now use the result from the previous question ($B_\nu(T) = 2h\nu^3/c^2 \exp\left(-h\nu/kT\right)$) to derive the Wien displacement law in terms of frequency (Note: last week we also looked at Wien's displacement law, but in terms of wavelength. After exercise 1, we can realise this is slightly different.)

Solution: In order to derive Wien's displacement law in terms of frequency, we now take the derivative of this formula and require it to be zero:

$$\frac{\partial B_\nu(T)}{\partial \nu} = \frac{6h\nu^2}{c^2}\exp\left(-h\nu/kT\right) - \frac{h}{kT}\frac{2h\nu^3}{c^2}\exp\left(-h\nu/kT\right) = 0.$$

Simplifying gives:

$$\nu_{\max} = \frac{3k}{h}T = 6.25109 \times 10^{10} T.$$

(Note: a full derivation without using the approximation of Wien, results in $\nu_{\max} = 2.82kT/h$, so the approximation is correct to within $\sim 6\%$.)

(c) Finally calculate the frequency at which the Solar radiation peaks (remember the Solar temperature we calculated last week, was $T = 5778\,\mathrm{K}$).

Solution:
Using the formula from the previous question (with T $= 5778\,\mathrm{K}$) to achieve the frequency at which the Solar spectrum reaches a maximum, we get:

$$\nu_{\max} = 3.6 \times 10^{14}\,\mathrm{Hz} = 360\,\mathrm{THz},$$

which translates to a wavelength of 831 nm.

The fact that this peak value is quite different from the one calculated last week (in terms of wavelength) should not be surprising, because frequency and wavelength are not linearly related. This means that the distribution gets transformed nonlinearly, which allows the peak of the distribution to shift significantly. This is an important (though often overlooked) aspect of non-linear parameter transformations and is more widely relevant, for example in the conversion of parallax to distance.

3. Earlier, we discussed how Kepler's law can be used to derive the masses of planets, assuming the mass of their moons are negligible in comparison. In the case of binary stars, this assumption is typically not correct. However, since both stars in such a system move symmetrically (weighted by their repective masses) around the common centre of gravity, it is easily realised that their mass ratio equals the inverse ratio of their orbital sizes: $m_1/m_2 = a_2/a_1$. Combining this equation with Kepler's third law ($P^2 M_{\mathrm{tot}} = a^3$, in which a is the sum of the semi-major axes of the two companion stars), we can disentangle the masses from both stars.

Illustrate the above by calculating the masses for a binary pair of stars that orbit every 100 years and are at most $7''$ and at least $1''$ separated on the sky. Assume a distance (to Earth) of 10 pc; the heaviest star moves by $2''$ during its orbit.

Solution:
If the maximum separation is $7''$, the minimum separation is $1''$ and the smallest orbit is $2''$ across, then the larger orbit must be $6''$ across. So the semi-major axis is:

$$a = 6/2 + 2/2 = 4''.$$

Knowing that by definition a length of 1 AU at a distance of 1 parsec corresponds to an angular lenght of 1 arcsec, then this 4 arcsec angular length at a distance of 10 pc has to correspond to 40 AU.

Putting this into Kepler's equation gives:

$$m_1 + m_2 = \frac{40^3}{100^2} \mathrm{M}_\odot = 6.4 \, \mathrm{M}_\odot.$$

Now given that $m_1/m_2 = a_2/a_1 = 3$ thus $m_1 = 3m_2$. Hence: $4m_2 = 6.4 \, \mathrm{M}_\odot$ thus:

$$m_2 = 1.6 \, \mathrm{M}_\odot$$

and $m_1 = 4.8 \, \mathrm{M}_\odot$.