Introduction to Astronomy
Exercises week 5

8 November 2019

1. The Solar flux at the Earth (remember $1\,\text{AU} = 1.495978 \times 10^{11}\,\text{m}$), is $1370\,\text{W/m}^2$. By integrating Planck's law (i.e. the blackbody spectrum), we can achieve Stefan-Boltzmann's law, which states that the frequency-integrated power output per unit surface area of a blackbody scales with temperature as: $M = \sigma T^4$, where $\sigma = 5.6705 \times 10^{-8}\,\text{Wm}^{-2}\text{K}^{-4}$ is the Stefan-Boltzmann constant and M is the total energy flux per unit surface area and unit time.

   (a) Calculate the total power output of the Sun and the Sun's temperature, assuming it is a blackbody. (Note the Solar radius is about 696,342 km.)

   Solution:
   The total power output is simply derived, if we assume that the output is fully isotropic:

   $$F = 1370\,\text{W/m}^2 \times 4\pi\,(1\,\text{AU})^2 = 1370 \times 4\pi \times \left(1.495978 \times 10^{11}\right)^2\,\text{W} = 3.85 \times 10^{26}\,\text{W}.$$

   Therefore the energy flux per unit surface area, is:

   $$M = F/A = F/\left(4\pi \times 696342000^2\right) = 6.32 \times 10^7\,\text{W/m}^2.$$

   We can now use Stefan-Boltzmann's law to determine the temperature:

   $$T = (M/\sigma)^{1/4} = 5778\,\text{K}.$$

   (b) By differentiating Planck's law, we can find the frequency at which the luminosity of a blackbody peaks. This is called Wien's law and states: $\lambda_{\text{peak}} = hc/(\beta k T)$ with $\beta = 4.96511$, $k = 1.380658 \times 10^{-23}\,\text{J/K}$ Boltzmann's constant and $h = 6.626 \times 10^{-34}\,\text{Js}$ Planck's constant. Now calculate the peak wavelength and frequency for the Solar spectrum.

   Solution:
   With a temperature of 5778 K, we have the peak brightness at:

   $$\lambda_{\text{peak}} = \frac{6.626 \times 10^{-34} \times 3 \times 10^8}{4.96511 \times 1.380658 \times 10^{-23} \times 5778}\,\text{m} = 502 \times 10^{-9}\,\text{m}$$

   or 502 nm. This equates to $\nu = c/\lambda = 598\,\text{THz}$.

   (c) Using the numbers above, estimate the amount of mass-loss this power output equates with, per second. At this rate, how long does the Sun survive? (The Sun's weight is approximately $2 \times 10^{30}\,\text{kg}$.)

   Solution:
   Using $E = mc^2$ and the total power ouput from question a), we get:

   $$m_{\text{persec}} = E_{\text{persec}}/c^2 = \frac{3.85 \times 10^{26}\,\text{W}}{(3 \times 10^8\,\text{m/s})^2} = 4.28 \times 10^9\,\text{kg/s}.$$

   At this rate, the Sun would survive for $2 \times 10^{30}/4.28 \times 10^9\,\text{s} = 4.67 \times 10^{20}\,\text{s}$ or about 14807 billion years. In practice, this amount of mass loss is roughly equal to that of the Solar wind and neither have any impact on the evolution of the Sun, because over the Sun's lifetime (of the order of ten billion years), it doesn't amount to a significant fraction of the Solar mass.

2. The sensitivity of radio telescopes is often expressed in terms of the Jansky (Jy), where $1\,\mathrm{Jy} = 10^{-26}\mathrm{W/m^2/Hz}$. Assuming a sensitivity of 1 mJy for a system that has a bandwidth of 300 MHz centred at 1.4 GHz, how far can you place the Sun before it becomes undetectable? Use the Rayleigh-Jeans approximation to the Planck spectrum, for long wavelengths: $I_\nu = 2kT\nu^2/c^2$. Given that $1\,\mathrm{AU} = 4.85\,\mu\mathrm{pc}$ and with a Galactic radius of $\sim 15\,\mathrm{kpc}$, in what fraction of the Galaxy can we see Sun-like stars in the radio?

Solution:

We found the temperature of the Sun to be 5778 K, therefore, the intensity of the Sun at 1.4 GHz is:

$$I = 2 \times 1.38 \times 10^{-23}\,\mathrm{J/K} \times 5778\,\mathrm{K} \times \left(\frac{1.4 \times 10^9\,\mathrm{Hz}}{3 \times 10^8\,\mathrm{m/s}}\right)^2 = 3.5 \times 10^{-18}\,\mathrm{W/m^2/Hz} = 3.5 \times 10^8\,\mathrm{Jy}.$$

Now, this intensity scales with the inverse of distance squared (and it is given at the Solar surface), so: $I_{\mathrm{lim}} \times D_{\mathrm{lim}}^2 = I \times R_\odot^2$. Hence, the distance for a limiting flux of 1 mJy is:

$$D_{\mathrm{lim}}/R_\odot = \sqrt{I/I_{\mathrm{lim}}} = \sqrt{3.5 \times 10^8/10^{-3}} = 5.9 \times 10^5.$$

So we can see Sun-like stars in the radio out to 0.59 million solar radii. Since $R_\odot = 696342 \times 10^3\,\mathrm{km}$, $1\,\mathrm{AU} = 1.495978 \times 10^{11}\,\mathrm{m}$ and $1\,\mathrm{AU} = 4.85\,\mu\mathrm{pc}$, we have: $R_\odot = 2.258 \times 10^{-8}\,\mathrm{pc}$, or: $D_{\mathrm{lim}} = 5.9 \times 10^5 \times 2.258 \times 10^{-8} = 0.013\,\mathrm{pc}$.

In terms of fraction of the Galaxy, this is $0.013^2/15000^2 = 7.9 \times 10^{-13}$ or $7.9 \times 10^{-11}\%$ of the Galactic disk.

In practice, the nearest star is proxima centauri, at a distance of 1.3 pc, which puts it much too far away to be observable with the system described here. This is the main reason why stellar astronomy is never performed at radio wavelengths (except in the case of the Sun).