Introduction to Astronomy
Exercises Week 3

25 October 2019

1. When the Earth is in perigee (147098 Mm from the Sun), the Sun has an angular diameter of $32.5'$. Assuming the Sun is spherical, and using Kepler's third law $T^2/r^3 = 4\pi^2/(GM)$ with $G = 6.67259 \times 10^{-11}\,\mathrm{Nm^2/kg^2}$ Newton's gravitational constant, determine the density of the Sun. ($1\,\mathrm{AU} = 1.496 \times 10^{11}\,\mathrm{m}$.)

> **Solution:** An angular diameter of $32.5'$ corresponds to:
>
> $$32.5/60 \times \pi/180 = 0.00945\,\mathrm{rad}.$$
>
> At a distance of 147098 Mm, this corresponds to a spatial diameter of:
>
> $$d = 0.00945 \times 147098 \times 10^6\,\mathrm{m} = 1.4 \times 10^9\,\mathrm{m}.$$
>
> Given that the Earth revolves around the Sun in 365.25 days ($= 3.15576 \times 10^7$ sec), with a semi-major axis of 1 AU, we can use Kepler's third law, which gives:
>
> $$M = \frac{r^3 4\pi^2}{GT^2} = \frac{\left(1.496 \times 10^{11}\,\mathrm{m}\right)^3 \times 4\pi^2}{6.67259 \times 10^{-11}\,\mathrm{Nm^2/kg}^2 \times \left(3.15576 \times 10^7\,\mathrm{s}\right)^2} = 1.9 \times 10^{30}\,\mathrm{kg}$$
>
> So the density is then:
>
> $$\rho = M/V = \frac{1.9 \times 10^{30}\,\mathrm{kg}}{d^3\pi/6} = \frac{12.0 \times 10^{30}\,\mathrm{kg}}{\pi\left(1.4 \times 10^9\,\mathrm{m}\right)^3} = 1.3 \times 10^3\,\mathrm{kg/m^3}$$
>
> (Note we calculated the volume as $4\pi r^3/3 = \pi d^3/6$ where r is the radius and d is the diameter.)

2. For an interstellar cloud to gravitationally collapse, it must be beyond its *Jeans mass*, which is roughly defined as:

$$M_{\mathrm{Jeans,M_\odot}} \approx 3 \times 10^4 \sqrt{T_{\mathrm{K}}^3/n_{\mathrm{atoms/m^3}}}.$$

For a typical interstellar molecular cloud, with a temperature of 20 K and a typical density of 1000 atoms/cm^3, calculate the required mass for gravitational collapse.

> **Solution:**
> The Jeans mass is:
>
> $$M \approx 3 \times 10^4 \sqrt{\frac{20^3}{1000 \times 10^6}}\, M_\odot = 84.9\,\mathrm{M_\odot}.$$
>
> Given that the Sun is quite a typical star, it should be clear that it's highly unlikely for a Solar-mass star to form on its own. In other words, this clarifies why stars typically form in groups ("clusters") and why binary stars are so common, because when a cloud collapses, it is typically large and massive enough to form many stars, not just one.
>
> Also note that the warmer and more tenuous this cloud is, the more massive it needs to be before it manages to collapse. This explains why only the coldest, densest clouds are important for star formation.

3. Using conservation of energy and conservation of angular momentum, derive the vis-viva relation (conservation of orbital energy):

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right).$$

Solution:
We know the kinetic energy is: $E_{\mathrm{kin}} = 0.5mv^2$ and potential energy is: $E_{\mathrm{pot}} = -GMm/r$. Hence, total energy is:

$$E_{\mathrm{Tot}} = \frac{v^2 m}{2} - \frac{GMm}{r} = \mathrm{const.}$$

Now, conservation of angular momentum gives $h = rv = \mathrm{const.}$ Using this at the apogee and perigee, where r is easily determined, we get:

$$\frac{v_{\mathrm{ap}}^2}{2} - \frac{GM}{r_{\mathrm{ap}}} = \frac{v_{\mathrm{peri}}^2}{2} - \frac{GM}{r_{\mathrm{peri}}}$$

and

$$v_{\mathrm{peri}} = \frac{r_{\mathrm{ap}}}{r_{\mathrm{peri}}} v_{\mathrm{ap}}.$$

Hence:

$$\frac{v_{\mathrm{ap}}^2}{2} - \frac{GM}{r_{\mathrm{ap}}} = \frac{v_{\mathrm{ap}}^2}{2}\frac{r_{\mathrm{ap}}^2}{r_{\mathrm{peri}}^2} - \frac{GM}{r_{\mathrm{peri}}},$$

or:

$$\frac{v_{\mathrm{ap}}^2}{2}\left(1 - \frac{r_{\mathrm{ap}}^2}{r_{\mathrm{peri}}^2}\right) = GM\left(\frac{1}{r_{\mathrm{ap}}} - \frac{1}{r_{\mathrm{peri}}}\right)$$

$$\frac{v_{\mathrm{ap}}^2}{2} = GM\left(\frac{r_{\mathrm{peri}} - r_{\mathrm{ap}}}{r_{\mathrm{ap}} r_{\mathrm{peri}}}\right)\left(\frac{r_{\mathrm{peri}}^2}{r_{\mathrm{peri}}^2 - r_{\mathrm{ap}}^2}\right)$$

$$\frac{v_{\mathrm{ap}}^2}{2} = GM\frac{r_{\mathrm{peri}} - r_{\mathrm{ap}}}{r_{\mathrm{peri}}^2 - r_{\mathrm{ap}}^2}\frac{r_{\mathrm{peri}}}{r_{\mathrm{ap}}}.$$

now, since $r_{\mathrm{peri}}^2 - r_{\mathrm{ap}}^2 = (r_{\mathrm{peri}} - r_{\mathrm{ap}})(r_{\mathrm{peri}} + r_{\mathrm{ap}})$ and since $(r_{\mathrm{peri}} + r_{\mathrm{ap}})/2 = a$ with a the semi-maj. axis of the orbit, we get:

$$\frac{v_{\mathrm{ap}}^2}{2} = \frac{GM}{2a}\frac{r_{\mathrm{peri}}}{r_{\mathrm{ap}}}.$$

Entering this into the original equation, we get:

$$\frac{E_{\mathrm{Tot}}}{m} = \frac{GM}{2a}\frac{r_{\mathrm{peri}}}{r_{\mathrm{ap}}} - \frac{GM}{r_{\mathrm{ap}}} = \frac{GM}{2a}\left(\frac{r_{\mathrm{peri}} - 2a}{r_{\mathrm{ap}}}\right).$$

Using again that $(r_{\mathrm{peri}} + r_{\mathrm{ap}})/2 = a$ and therefore that $r_{\mathrm{peri}} - 2a = -r_{\mathrm{ap}}$, we have:

$$\frac{E_{\mathrm{Tot}}}{m} = -\frac{GM}{2a}.$$

The total energy can hence be re-written for the general case:

$$\frac{v^2}{2} - \frac{GM}{r} = -\frac{GM}{2a}$$

or:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right),$$

which is the desired vis-viva relation.