# Introduction to Astronomy
# Exercises Week 2

18 October 2019

1. The resolution of telescopes can be approximated as:

$$\theta \approx 1.22 \frac{\lambda}{D}$$

where $\theta$ is the resolution in radians, $\lambda$ is the observing wavelength in metres and $D$ is the aperture diameter in metres.

(a) Considering the 10-m Keck telescopes operating at a wavelength of 580 nm (in the yellow part of the spectrum), calculate the minimum angular diameter (in radians and arcseconds) a source will have to be in order to be resolved.

**Solution:**
The minimum angular diameter in radians is easy:

$$\theta \approx 1.22 \frac{580 \times 10^{-9}}{10} = 7.076 \times 10^{-8} \text{ radians.}$$

Recalculating into arcseconds, we get:

$$\theta \approx 7.076 \times 10^{-8} \text{ rad} \times \frac{180}{\pi} * 3600 = 15 \text{ mas}$$

So the resolution is just better than 15 mas, which means that objects will have to be larger than 15 mas in order to be resolved.

In practice this high resolution does not get achieved, though, since various corrupting effects play a role. Most importantly *seeing*, where instabilities (turbulence) in the Earth's atmosphere cause the object to "wander around", thereby smearing the image. A more realistic estimate of resolution that is practically achievable on Earth, is about an order of magnitude higher.

(b) With interferometric techniques, multiple telescopes can be used in combination to synthesise a large one. In that case the aperture diameter in the above equation becomes equal to the maximal spacing between telescopes.

Calculate the resolution of the very large array – a 27-element interferometer in New Mexico – operating at an observing frequency of 50 GHz and in its most extended configuration, with a maximal baseline length of 36 km.

**Solution:**
First, we calculate the observing wavelength:

$$\lambda = c/\nu = \frac{3 \times 10^{8} \text{m/s}}{50 \times 10^{9} \text{Hz}} = 0.006 \text{ m.}$$

Now the resolution follows trivially:

$$\theta \approx 1.22 \frac{0.006}{36 \times 10^{3}} = 2 \times 10^{-7} \text{ radians} = 42 \text{ mas.}$$

Because radio waves have much longer frequencies, they are less affected by atmospheric turbulence, meaning that this angular resolution is actually realistically achievable.

2. (a) The orbital period of Mars around the Sun is 1.8808 years. Using Kepler's third law ($a^3_{\mathrm{AU}} = (M_{1,M_\odot} + M_{2,M_\odot})P^2_{\mathrm{yrs}}$), calculate the semi-major axis of the Martian orbit.

> **Solution:**
> The semi-major axis follows straightforwardly, assuming $M_{\mathrm{Mars}} \ll M_{\mathrm{Sun}}$:
>
> $$a_{\mathrm{AU}} = \left(1 \times 1.8808^2\right)^{1/3} = 1.52\,\mathrm{AU}.$$

(b) Using the result from the previous question and the fact that the astronomical unit equals $1.496 \times 10^8$ km, estimate the radius of Mars if its angular radius is observed to be 9.2″ at opposition (when the Earth is directly between Mars and the Sun). Approximate orbits as perfectly circular.

> **Solution:** When the Earth is directly between Mars and the Sun; and approximating the Earth's orbit as a circle, we have a distance between the Earth and Mars of 0.52 AU. From trigonometry, it follows that the angular radius $\alpha$ is related to the radius $r$ of and the distance $D$ to the object as: $\alpha = \mathrm{asin}(r/D)$, or, more simply, as: $\alpha = r/D$. Hence, we get:
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> $$r = \alpha d = 9.2''0.52\,\mathrm{AU} = 9.2/3600/180 \times \pi \times 0.52 \times 1.496 \times 10^{11}\,\mathrm{m} = 3.470 \times 10^6\,\mathrm{m}.$$
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> So the radius of Mars must be about 3470 km.

(c) Now that you know how far Mars is, you can calculate its mass by observing the orbit of its moon. Specifically, Phobos orbits Mars in 0.3189 days and when Mars is in opposition, the maximum separation between Phobos and Mars is seen to be 25″. What is Mars' mass (compared to Earth)? (Assume a circular orbit for Phobos; and use a solar mass of $\mathrm{M}_\odot = 332968 \times \mathrm{M}_{\mathrm{Earth}}$.)

> **Solution:** The distance between Mars and Phobos follows from geometry:
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> $$r = \alpha \times d = 25'' \times 0.52\,\mathrm{AU} = 25/3600/180 \times \pi \times 0.52\,\mathrm{AU} = 6.30 \times 10^{-5}\,\mathrm{AU} = 9429\,\mathrm{km}.$$
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> Now, assuming $M_{\mathrm{Mars}} \gg M_{\mathrm{Phobos}}$, we can use Kepler's third law again:
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> $$M_{\mathrm{Mars}} = \frac{a^3}{P^2} = \frac{(6.30 \times 10^{-5})^3}{(0.3189/365.25)^2}\mathrm{M}_\odot = 3.28 \times 10^{-7}\mathrm{M}_\odot = 0.109\mathrm{M}_{\mathrm{Earth}}.$$

3. There are two obvious definitions of a day: the sidereal day, after which the Earth has regained its orientation with respect to the stars; and the Solar day, after which the Earth has regained its orientation with respect to the Sun. Calculate the difference between the two; and how much does this difference add up to in a year? (Note that there are 86400 seconds in a Solar day and 365.25 days in a year, *by definition.*)

> **Solution:** The Solar day lasts 24 hours. Since it takes Earth 365.25 days to revolve around the Sun, in those 24 hours, the Earth has travelled $2\pi/365.25$ radians in its orbit around the Sun. This means that during a Solar day, the Earth has rotated $2\pi(1 + 1/365.25)$ radians around its axis, instead of just $2\pi$. In other words:
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> $$\frac{T_{\mathrm{Stellar}}}{T_{\mathrm{Solar}}} = \frac{2\pi}{2\pi(1 + 1/365.25)}.$$
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> Now, with $T_{\mathrm{Solar}} = 86400$ sec, we get:
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> $$T_{\mathrm{Stellar}} = \frac{1}{1 + 1/365.25}86400\,\mathrm{sec} = 86164\,\mathrm{sec}.$$
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> This is 236 seconds or $3^{\mathrm{m}} : 56^{\mathrm{s}}$ less than a Solar day. So the Earth actually rotates around its axis once every $23^{\mathrm{h}}56^{\mathrm{m}}4^{\mathrm{s}}$, instead of every 24 hours.
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> Over the course of a year, this mismatch results in $365.25 * 236$ s= 86199 sec or nearly a day. So there is essentially one less solar day in the year than there are sidereal days.

An alternative – and possibly easier and shorter, but equally correct – solution would be to point out that over the course of a year, there must be exactly one more sidereal day than solar day (to make up for the single rotation the Earth has made around the Sun). This means that the fractional difference in length would be:

$$\frac{T_{\mathrm{Stellar}}}{T_{\mathrm{Solar}}} = \frac{366.25}{365.25} = 1.00274,$$

from which it follows that a sidereal day must be 86400/1.00274 seconds long.