Kettenregel :

$$\underline{\underline{\left[f(g(x))\right]' = f'(g(x)) \cdot g'(x)}}$$

↑ „äussere Abl."

↑ „nachdifferenzieren" / „innere Abl."

Strenger Bew: Analysis

Anschauliche Begründung:

$$\left[f(g(x_0))\right]' = \lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0} \cdot \underbrace{\frac{g(x) - g(x_0)}{g(x) - g(x_0)}}_{=1}$$

$$= \lim_{x \to x_0} \underbrace{\frac{f(g(x)) - f(g(x_0))}{\underbrace{g(x)}_{=:g} - \underbrace{g(x_0)}_{=:g_0}}}_{\to \lim\limits_{g \to g_0} \frac{f(g) - f(g_0)}{g - g_0} = f'(g_0) = f'(g(x_0))} \cdot \underbrace{\frac{g(x) - g(x_0)}{x - x_0}}_{\to g'(x_0)} = f'(g(x_0)) \cdot g'(x_0)$$

Noch einfacher: $\frac{d f(g(x))}{dx} = \frac{d f(g(x))}{d g(x)} \frac{d g(x)}{dx}$ oder $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

Beisp:

1.) $\underline{\underline{(\sin(2x))'}} \overset{f(x)=\sin x,\ g(x)=2x}{=} f'(g(x)) \cdot g'(x) = \cos(2x)\cdot 2 = \underline{\underline{2\cos(2x)}}$

2.) Analog: $\underline{\underline{(f(bx))'}} \overset{g(x)=bx}{=} f'(bx)\cdot \overbrace{g'(x)}^{b} = \underline{\underline{b\cdot f'(bx)}}$

3.) $f(x) := x^n, \quad g(x) := a+bx$

$\Rightarrow \quad f(g(x)) = (a+bx)^n$

$\Rightarrow \quad \underline{\underline{\frac{d\,(a+bx)^n}{dx}}} = \underline{\underline{\underbrace{n(a+bx)^{n-1}}_{f'(g(x))} \cdot \underbrace{b}_{g'(x)}}}$

4.) $g(x) := f^{-1}(x) \Rightarrow f(g(x)) = f(f^{-1}(x)) = x$

$$\Rightarrow \underbrace{\left(\underbrace{f(f^{-1}(x))}_{x}\right)'}_{x' = 1} = f'(f^{-1}(x)) \cdot (f^{-1}(x))'$$

$$\Rightarrow (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

5.) $f(x) := \exp(x) \Rightarrow f^{-1}(x) = \ln(x) \quad (x \in \mathbb{R}^+)$, $f'(x) = \exp(x)$

$$\Rightarrow (\ln x)' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\underbrace{\exp(\ln(x))}_{=x}} = \frac{1}{x}$$

$$\frac{d \ln(x)}{dx} = \frac{1}{x} \quad \forall x \in \mathbb{R}^+$$

8.4 Höhere Ableitungen

$$\frac{d^2 f(x)}{dx^2} = \frac{d^2}{dx^2} f(x) := \frac{d}{dx}\left(\frac{d}{dx} f(x)\right) \quad \text{usw:}$$

$$\frac{d^n f(x)}{dx^n} := \frac{d}{dx}\left(\frac{d^{n-1} f(x)}{dx^{n-1}}\right) \qquad \text{„n-te Ableitung"}$$

Andere Schreibweisen:

$$f'(x),\ f''(x),\ f'''(x),\ f^{(4)}(x),\ \ldots,\ f^{(n)}(x) \quad (n \in \mathbb{N}_0)$$

bzw.

$$\dot{f}(t),\ \ddot{f}(t),\ \dddot{f}(t),\ \underset{\substack{\uparrow \\ \text{ab } n=4}}{f^{(n)}(t)}$$

Beisp:

1.) $\sin'(x) = \cos(x)$

$\sin''(x) = \cos'(x) = -\sin(x)$

$\sin'''(x) = -\sin'(x) = -\cos(x)$

$\sin^{(4)}(x) = -\cos'(x) = \sin(x)$ usw.

$\sin^{(n+4)}(x) = \sin^{(n)}(x) \quad \forall\, n \in \mathbb{N}_0$

ebenso für $\cos(x)$

2.) $f(t) := \ln t \Rightarrow \dot{f}(t) = \frac{1}{t}, \quad \ddot{f}(t) = \frac{d}{dt}\left(\frac{1}{t}\right) = -\frac{1}{t^2}$

8.5 Ableitung vektorwertiger Funktionen

Vektorwertige Fkt'en aus Kap. 4:

$$\vec{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix} = \sum_{k=1}^{n} f_k(t)\,\vec{e}_k \qquad , \; n \in \mathbb{N}$$

Def:

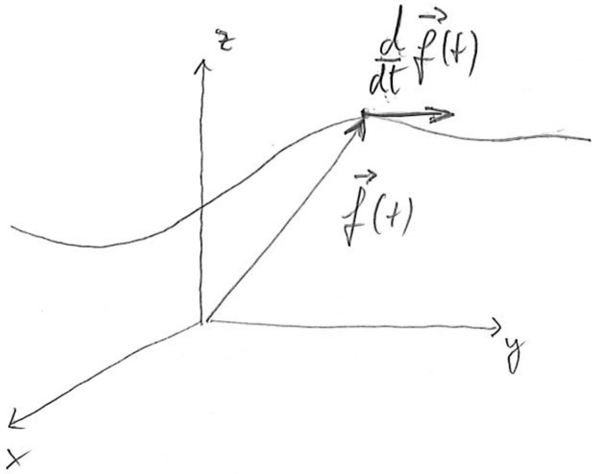
$$\frac{d}{dt}\vec{f}(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \underbrace{\left(\vec{f}(t+\Delta t) - \vec{f}(t)\right)}_{\sum_{k=1}^{n} \left(f_k(t+\Delta t) - f_k(t)\right)\vec{e}_k} =$$

$$= \lim_{\Delta t \to 0} \sum_{k=1}^{n} \underbrace{\frac{f_k(t+\Delta t) - f_k(t)}{\Delta t}}_{\to \dot{f}_k(t)} \vec{e}_k \qquad \Leftrightarrow$$

$$\frac{d}{dt}\vec{f}(t) = \begin{pmatrix} \dot{f}_1(t) \\ \vdots \\ \dot{f}_n(t) \end{pmatrix} = \sum_{k=1}^{n} \dot{f}_k(t)\,\vec{e}_k$$

(„komponentenweise")

Veranschaulichung im $\mathbb{R}^3$:

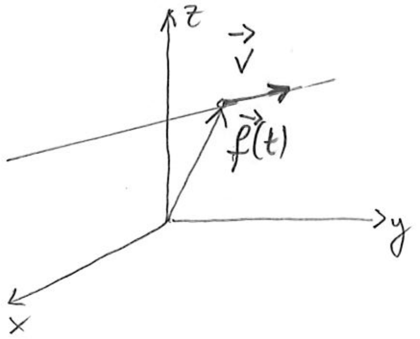


z.B. $\vec{f}(t)$ „Ort" $\Rightarrow$ $\frac{d}{dt}\vec{f}(t)$ „Geschw."

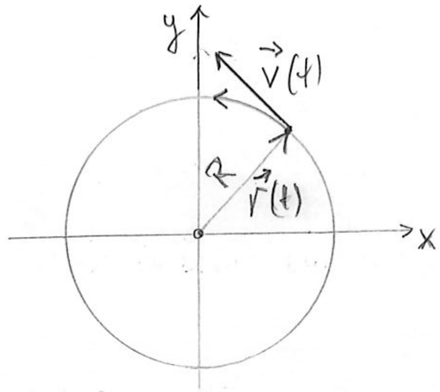
Beisp: [genau wie in Kap. 4]

1.) $\vec{f}(t) := \vec{r_0} + t\,\vec{v} = \begin{pmatrix} r_{01} + tv_1 \\ r_{02} + tv_2 \\ r_{03} + tv_3 \end{pmatrix}$ (Ort)

↳ Startpkt.



$\frac{d}{dt}\vec{f}(t) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{v}$ (konstante Geschw.)



2.) $\vec{r}(t) := \begin{pmatrix} R\cos(\omega t) \\ R\sin(\omega t) \end{pmatrix}$

„Ort", $R, \omega > 0$

$\Rightarrow \underline{\underline{|\vec{r}(t)|}} = \sqrt{R^2\cos^2(\omega t) + R^2\sin^2(\omega t)} = \sqrt{R^2} \underset{R>0}{=} \underline{\underline{R}}$

$\Rightarrow \vec{v}(t) := \frac{d}{dt}\vec{r}(t)$ („Geschw.")

$= \begin{pmatrix} R(-\sin(\omega t)\cdot\omega) \\ R(\cos(\omega t)\cdot\omega) \end{pmatrix} = R\omega \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{pmatrix}$

$\Rightarrow \underline{\underline{|\vec{v}(t)|}} = \sqrt{R^2\omega^2\sin^2(\omega t) + R^2\omega^2\cos^2(\omega t)} = \underline{\underline{R\omega}} = \omega\cdot|\vec{r}(t)|$ $(\omega \geq 0)$

$$\Rightarrow \quad \vec{r}(t)\cdot\vec{v}(t) = R\begin{pmatrix}\cos(\omega t)\\ \sin(\omega t)\end{pmatrix}\cdot R\omega\begin{pmatrix}-\sin(\omega t)\\ \cos(\omega t)\end{pmatrix}$$

$$= R^2\omega\left(\cos(\omega t)(-\sin(\omega t)) + \sin(\omega t)\cos(\omega t)\right) = 0$$

$$\Rightarrow \quad \underline{\underline{\vec{v}(t)\perp\vec{r}(t)}} \text{ (senkrecht) } \forall t \,!$$

$$\Rightarrow \quad \underline{\underline{\vec{a}(t)}} := \frac{d}{dt}\vec{v}(t) = \frac{d^2}{dt^2}\vec{r}(t) \qquad (\text{„Beschl."})$$

$$= R\omega\begin{pmatrix}-\cos(\omega t)\cdot\omega\\ -\sin(\omega t)\cdot\omega\end{pmatrix} = \underline{\underline{-\omega^2\vec{r}(t)}} \qquad \Rightarrow \underline{\underline{\vec{a}(t)\perp\vec{v}(t)}}$$

$$\Rightarrow \underline{\underline{|\vec{a}(t)|}} = \omega^2|\vec{r}(t)| = \underline{\underline{\omega^2 R}} = \omega|\vec{v}(t)| = \underline{\underline{|\vec{v}(t)|^2/R}}$$

Genauso im $\mathbb{R}^3$: $\quad \vec{r}(t) := \begin{pmatrix}R\cos(\omega t)\\ R\sin(\omega t)\\ 0\end{pmatrix}$

$$\Rightarrow \quad \vec{v}(t)\perp\vec{r}(t)$$

$$\vec{a}(t) = -\omega^2\vec{r}(t) \quad \perp \vec{v}(t) \qquad \text{usw.}$$

Vgl. EP 1 (Th. Huser)