Quantum Mechanics: Exercises 10

Due to: January 15, 2013.

Problem 1

In classical physics, a vector is quantity which transforms like $V_i \to R_{ij}V_j$ under rotations. In quantum mechanics, a vector operator $\hat{V}_i$ is a quantity whose expectation value transforms like a vector, i.e. if state transforms like

$$|\psi\rangle \to \hat{U}(\boldsymbol{n}, \alpha)|\psi\rangle, \tag{1}$$

then expectation value of vector operator transforms like

$$\left\langle \psi \middle| \hat{V}_i \middle| \psi \right\rangle \to \left\langle \psi \middle| \hat{U}^\dagger(\boldsymbol{n}, \alpha)\hat{V}_i\hat{U}(\boldsymbol{n}, \alpha) \middle| \psi \right\rangle = R_{ij}\left\langle \psi \middle| \hat{V}_j \middle| \psi \right\rangle, \tag{2}$$

where R_{ij} is a matrix that corresponds to rotation around axis $\boldsymbol{n}$ by angle α. Use this relation for infinitesimal rotations and prove that vector operator $\hat{V}_i$ satisfies commutation relation

$$[\hat{V}_i, \hat{J}_j] = i\epsilon_{ijk}\hbar\hat{V}_k. \tag{3}$$

Show then that a coordinate and a momentum are vector operators and that every vector operators satisfies

$$\left[\hat{\vec{J}}^2, \left[\hat{\vec{J}}^2, \hat{V}_i\right]\right] = 2\hbar^2\left(\hat{\vec{J}}^2\hat{V}_i + \hat{V}_i\hat{\vec{J}}^2\right) - 4\hbar^2\hat{J}_i\left(\hat{\vec{J}} \cdot \hat{\vec{V}}\right). \tag{4}$$

Problem 2

On lecture it was stated that term proportional to the $\boldsymbol{B}^2$ can be neglected. However, this term leads to the quadratic Zeeman effect. Calculate energy shift to first order of perturbation theory of the ground state of hydrogen atom due to the $e^2\boldsymbol{A}^2/(2m)$ term in the Hamiltonian. Write the energy shift as

$$\Delta E = -\frac{1}{2}\chi\boldsymbol{B}^2, \tag{5}$$

and obtain expression for diamagnetic susceptibility, χ.