Quantum Mechanics: Exercises 1

Due to: October 30, 2012.

Problem 1

Show that any 2x2 matrix can be expressed as a combination of the I_2 (2x2 unity matrix) and so-called Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

Prove their commutation and anti-commutation relations:

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \tag{2}$$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}.I_2, \tag{3}$$

where ϵ_{ijk} is the Levi-Civita symbol.

Problem 2

Operator is said to be Hermitian if for its matrix elements holds that $M_{ij} = M^*_{ji}$. Show then that matrix

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{4}$$

a) is non-Hermitian.
b) it has real eigenvalues, but its eigenvectors do not form complete set.
c) find a vector $|v\rangle$ such that $\langle v|M|v\rangle$ is complex.
d) Why non-Hermitian operators cannot represent physical observables?

Problem 3

With knowledge that polarization states of electrons can be represented by

$$|x\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}, \quad |y\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}, \quad |z\pm\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5}$$

prove that matrix elements of the operator $\hat{\mu}_i$ are given by

$$\hat{\mu}_i = \mu_0 \sigma_i, \tag{6}$$

where μ_0 is some constant and σ_i are Pauli matrices.

Problem 4

Prove that

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB. \tag{7}$$