Tutorial sheet 9

32. Symmetrizer and antisymmetrizer

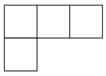
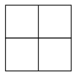
In the lecture on December 1st, the symmetrizer and antisymmetrizer

$$\mathscr{S} \equiv \frac{1}{n!}\sum_{\sigma\in\mathrm{S}_n}\sigma \quad\text{and}\quad \mathscr{A} \equiv \frac{1}{n!}\sum_{\sigma\in\mathrm{S}_n}\varepsilon(\sigma)\sigma\,, \tag{1}$$

elements of the group algebra $\mathbb{C}\mathrm{S}_n$ of the symmetric group S_n, were introduced.
Let $\tau \in \mathrm{S}_n$ be a permutation. Compute first $\tau\mathscr{S}$ and $\tau\mathscr{A}$, and then $\mathscr{S}^2$, $\mathscr{A}^2$, $\mathscr{S}\mathscr{A}$ and $\mathscr{A}\mathscr{S}$.

33. Representations of S_n

The following Young diagrams are associated to two irreps. of the symmetric group S_4:

 ,  . (2)

Compute their outer product and give the dimensions of all irreducible representations involved.

34. Symplectic group

Let $\mathbb{1}_n$ denote the $n \times n$ identity matrix, where $n \in \mathbb{N}^*$.

i. Show that the set of $2n \times 2n$ matrices M with real entries satisfying

$$M^{\mathsf{T}}\begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}M = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \tag{3}$$

is a group — called *symplectic group* and denoted[1] $\mathrm{Sp}(2n,\mathbb{R})$ — for the usual matrix product. How many real parameters are needed to characterize an element of this group?
Hint: Find a relation between the inverse matrix M^{-1} and the transposed matrix M^{T}.

ii. Check that the matrices of $\mathrm{Sp}(2n,\mathbb{R})$ preserve the scalar product

$$\boldsymbol{x}\cdot\boldsymbol{y} \equiv \sum_{a=1}^{n}(x_a y_{a+n} - x_{a+n}y_a) \tag{4}$$

where $\boldsymbol{x},\boldsymbol{y}$ are $2n$-dimensional real vectors with coordinates x_a, y_a. Do you see where in physics the coordinate transformations realized by the symplectic matrices play a role? (*hint*: classical mechanics)

35. Exponential of a matrix

Let A and B be two complex $n \times n$ matrices. Show the following results:

i. If A and B commute, then $\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{A+B}$.

ii. $\left(\mathrm{e}^A\right)^{-1} = \mathrm{e}^{-A}$.

iii. If B is regular, then $B\,\mathrm{e}^A B^{-1} = \mathrm{e}^{BAB^{-1}}$.

iv. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then the eigenvalues of e^A are $\mathrm{e}^{\lambda_1}, \ldots, \mathrm{e}^{\lambda_n}$, so that $\det \mathrm{e}^A = \mathrm{e}^{\mathrm{Tr}\,A}$.

[1] The group is unfortunately sometimes denoted $\mathrm{Sp}(n,\mathbb{R})$...