Tutorial sheet 8

29. Tensor product of irreducible representations

Prove the following results regarding the (irreducible) representations of a finite group $\mathcal{G}$. In **ii.**, $\hat{\mathscr{D}}^*$ denotes the conjugate representation of $\hat{\mathscr{D}}$, as introduced in exercise **24.**

i. The tensor product of an irreducible representation with a representation of dimension 1 is irreducible.

ii. Let $\hat{\mathscr{D}}_1$, $\hat{\mathscr{D}}_2$ and $\hat{\mathscr{D}}_3$ be three unitary irreducible representations. The number of times that $\hat{\mathscr{D}}_1^*$ is contained in (the Clebsch–Gordan series of) $\hat{\mathscr{D}}_2 \otimes \hat{\mathscr{D}}_3$ is equal to the number of times that $\hat{\mathscr{D}}_2^*$ is contained in $\hat{\mathscr{D}}_1 \otimes \hat{\mathscr{D}}_3$ and to the number of times that $\hat{\mathscr{D}}_3^*$ is contained in $\hat{\mathscr{D}}_1 \otimes \hat{\mathscr{D}}_2$.

iii. a) Let $\hat{\mathscr{D}}_1$, $\hat{\mathscr{D}}_2$ be two unitary irreducible representations with respective dimensions d_1, d_2 with $d_1 \geq d_2$. The tensor product $\hat{\mathscr{D}}_1 \otimes \hat{\mathscr{D}}_2$ contains no representation of dimension lower than d_1/d_2.

Hint: The result from **ii.** can be helpful.

b) As we shall see later in the lecture, a "spin j" (particle) corresponds to an irreducible representation (of the group of rotations) of dimension $2j+1$. Refresh your knowledge from your Quantum Mechanics lecture on the addition of two spins j_1 and j_2. Check that the lower bound on the total spin which you learned back then is compatible with the result from **iii.a)**.

30. Crystal-field splitting

In the lecture of November 28th, you heard of the octahedral group $\mathcal{O}$ with its five inequivalent irreducible representations $\hat{\mathscr{D}}^{(1)}$, $\hat{\mathscr{D}}^{(1')}$, $\hat{\mathscr{D}}^{(2)}$, $\hat{\mathscr{D}}^{(3)}$, $\hat{\mathscr{D}}^{(3')}$: these represent the rotations leaving invariant the electric field felt by an ion at a site surrounded by a regular octahedron of identical ions in an ideal crystal. Suppose now that the crystal is distorted, and more precisely that it is elongated along one of the (four) axes with threefold symmetry.

i. Convince yourself that the distortion reduces the rotational symmetry of the crystal to the dihedral group D_3. Which rotations of $\mathcal{O}$ are still present in D_3?

Hint: The three rotations through 180° that survive are in the conjugacy class in $\mathcal{O}$ with 6 elements.

ii. With respect to the three irreducible representations of D_3, which you already encountered several times, the irreps. of $\mathcal{O}$ may become reducible. Perform those reductions and discuss their meaning for the degeneracy of states with angular-momentum quantum number $\ell = 0, 1, 2, 3$.

31. Conjugate representation (2)

The conjugate representation $\mathscr{D}^*$ to a matrix representation $\mathscr{D}$ of a (finite) group $\mathcal{G}$ was introduced in exercise **24.**. In this exercise, we want to classify the possible relations between $\mathscr{D}^*$ and $\mathscr{D}$.

i. Show that if the character $\chi_{\mathscr{D}}(g)$ of $\mathscr{D}(g)$ is real-valued for all $g \in \mathcal{G}$, then $\mathscr{D}$ and $\mathscr{D}^*$ are equivalent — in which case one talks of a *self-conjugate representation*.

ii. In exercise **24.ii.**, you showed that if $\mathscr{D}$ is an irreducible self-conjugate representation, then there exists $\lambda \in \mathbb{C}$ such that the similarity matrix C between every $\mathscr{D}(g)$ and $\mathscr{D}^*(g)$ obeys $CC^* = \lambda\mathbb{1}$. Show that you can choose $\lambda = \pm 1$.

iii. Assume further that $\mathscr{D}$ is unitary, so that you know from exercise **24.iii.** that the similarity matrix C can be taken to be either symmetric or antisymmetric: $C^{\mathrm{T}} = \pm C$.

a) Prove that if the character $\chi_{\mathscr{D}}$ is real-valued, then

$$\frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}}\chi_{\mathscr{D}}(g^2) = \pm 1, \tag{1}$$

the sign on the right hand side being the same as in the relation between C^{T} and C.

b) Show that if $\mathscr{D}$ is real, i.e. $\mathscr{D}^*(g) = \mathscr{D}(g)$ for all $g \in \mathcal{G}$, then Eq. (1) holds with $+1$ on the right hand side.

The converse is true: if the quantity on the left hand side of Eq. (1) equals $+1$, then $\mathscr{D}$ is real.[1]

iv. One can further prove that if $\mathscr{D}$ and $\mathscr{D}^*$ are not equivalent, then the character $\chi_{\mathscr{D}}$ is complex-valued and such that

$$\frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}}\chi_{\mathscr{D}}(g^2) = 0. \tag{2}$$

In exercise **25.** and **28.** you found the characters of the irreducible representations of the dihedral group D_4 and the cyclic group C_4. Compute the quantity on the left hand side of Eq. (1) / (2) for each irrep. of D_4 or C_4 and tell which ones are real, self-conjugate but not real ("pseudo-real"), or not self-conjugate ("complex").

[1] If you wish to prove it, show that there exists a unitary and symmetric matrix A such that $A^2 = C$ and consider the representation $\mathscr{D}' = A\mathscr{D}A^{-1}$.