Tutorial sheet 7

26. Regular representation

The *regular representation* of a finite group $\mathcal{G} = \{g_i\}$ is a $|\mathcal{G}|$-dimensional representation $\hat{\mathscr{D}}^{(\mathrm{r})}$ defined as follows. Given a basis $\{\mathrm{e}_i\}$ of the representation space $\mathscr{V}$, one first associates each group element g_i to the basis vector e_i for $i \in \{1, \ldots, |\mathcal{G}|\}$. For each $g \in \mathcal{G}$, the linear operator $\hat{\mathscr{D}}^{(\mathrm{r})}(g)$ on $\mathscr{V}$ maps the basis vector e_i to the basis vector e_j associated to the group element $g_j = gg_i$.[1]

i. Write down the matrices of the regular representations of the cyclic groups C_2, C_3 and C_4 and of the Klein group V_4.

ii. Determine the character $\chi^{(\mathrm{r})}$ of the regular representation.

Hint: Distinguish between the identity element e of $\mathcal{G}$ and the other elements of the group.

iii. Let $\{\hat{\mathscr{D}}^{(\alpha)}\}$ denote a set of inequivalent unitary irreducible representations of $\mathcal{G}$, with respective representation spaces $\{\mathscr{V}^{(\alpha)}\}$ and characters $\{\chi^{(\alpha)}\}$.

a) Show with the help of characters that the coefficient a_α in the decomposition $\hat{\mathscr{D}}^{(\mathrm{r})} = \bigoplus_\alpha a_\alpha \hat{\mathscr{D}}^{(\alpha)}$ of the regular representation equals the dimension of the representation space $\mathscr{V}^{(\alpha)}$.

b) Deduce from **a)** the equality $\sum_{\text{irreps. } \alpha} \left(\dim \mathscr{V}^{(\alpha)}\right)^2 = |\mathcal{G}|$.

27. Group representation from a representation of a quotient group

Let $\mathcal{N}$ be a normal subgroup of a group $\mathcal{G}$. A representation $\mathscr{D}_{\mathcal{G}/\mathcal{N}}$ of the quotient group $\mathcal{G}/\mathcal{N}$ can be *lifted* to a representation $\mathscr{D}_{\mathcal{G}}$ of the group $\mathcal{G}$ of the same dimension by defining

$$\mathscr{D}_{\mathcal{G}}(g) \equiv \mathscr{D}_{\mathcal{G}/\mathcal{N}}(g\mathcal{N}) \quad \forall g \in \mathcal{G}. \tag{1}$$

i. Check that Eq. (1) indeed defines a representation of $\mathcal{G}$. How are the characters of $\mathscr{D}_{\mathcal{G}/\mathcal{N}}$ and $\mathscr{D}_{\mathcal{G}}$ related?

ii. The symmetry group of a square D_4, which you already encountered in exercise **25.**, has a normal subgroup $\mathcal{N}$ consisting of the identity transformation and the rotation through 180° — this is actually the center of D_4.

a) Check that the quotient group $\mathrm{D}_4/\mathcal{N}$ is isomorphic to the Klein group V_4.

b) Write down the character table for V_4 and show that the characters of some of the irreducible representations of D_4 (exercise **25.**) can be lifted from those of the irreps. of V_4.

28. Symmetry group of a square (2)

The symmetry group of a square D_4 has a (normal) subgroup, consisting of the rotations that leave the square invariant, which is isomorphic to the group C_4.

i. What are the irreducible representations of this subgroup?

ii. In exercise **25.** you found that all one-dimensional (irreducible) representations of the group D_4 are real. This means that some of the irreps. of the normal subgroup C_4 are not irreps. of D_4. Explain why. What consequence can this have in a physical context?

[1] For the purist, the representation thus defined is the left regular representation of $\mathcal{G}$.