Tutorial sheet 6

22. Direct-product representation

Let $\mathcal{G}$ be a group and $\mathscr{D}$ an irreducible representation of $\mathcal{G}$. Show that $\mathscr{D} \otimes \mathscr{D}$ is an irreducible representation of the product group $\mathcal{G} \times \mathcal{G}$.

23. Möbius group

Consider the following nonlinear transformations in the complex plane:

$$z \to \frac{az+b}{cz+d}, \tag{1}$$

where a, b, c, d are complex numbers such that $ad \neq bc$.

i. Prove that the set of all such transformations forms a group (called the *Möbius group*) under composition. Why is the condition $ad \neq bc$ necessary?

ii. Explain why the following constraints define subgroups of the Möbius group:

a) $a = 1$, $b \in \mathbb{C}$, $c = 0$, $d = 1$; **b)** $|a| = 1$, $b = c = 0$, $d = 1$; **c)** $a \in \mathbb{R}^*$, $b = c = 0$, $d = 1$.

To which transformations in the complex plane do these subgroups correspond?

Remark: As opposed to these, the transformations with $c \neq 0$ do not preserve distance in the complex plane, yet they still preserve angles; such transformations are in general called conformal.

iii. Show that the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfy the same multiplication rule as the elements of the Möbius group. Do they provide a representation? Impose a suitable constraint on a, b, c, d such that resulting restricted set of matrices does provide a representation of the Möbius group.

Hint: The role of the constraint is to ensure that a given Möbius transformation maps to only one matrix.

24. Conjugate representation

Let $\mathscr{D}$ be a (matrix) representation of a group $\mathcal{G}$. For every $g \in \mathcal{G}$, $\mathscr{D}^*(g)$ denotes the complex conjugate of $\mathscr{D}(g)$.

i. Show that $\mathscr{D}^*$ is also a representation of $\mathcal{G}$.

ii. Assume that $\mathscr{D}$ and $\mathscr{D}^*$ are equivalent representations, i.e. there exists an invertible matrix C such that $\mathscr{D}^*(g) = C^{-1}\mathscr{D}(g)C$ for all $g \in \mathcal{G}$. Prove that if $\mathscr{D}$ is irreducible then there exists $\lambda \in \mathbb{C}$ such that $CC^* = \lambda \mathbb{1}$.

iii. Show that if $\mathscr{D}$ is unitary, then also $CC^\dagger = \mu \mathbb{1}$ with $\mu \in \mathbb{C}$. Moreover show that C may be chosen to be either symmetric or antisymmetric.

25. Symmetry group of a square

The symmetry group of a square is the dihedral group D_4. Determine its conjugacy classes and use them to find the number and dimensions of all irreducible representations of the group. Determine the character table of the group with the help of the orthogonality relations. Give the geometrical interpretation of all the irreducible representations.

Hint: The group D_4 has eight elements. You need to guess some of the representations in order to be able to use the orthogonality relations efficiently. The two one-dimensional and the two-dimensional representations are rather obvious.