Tutorial sheet 5

18. Faithful representation

Let $\mathcal{G}$ be a group and $\mathscr{D}$ a representation of $\mathcal{G}$. Show that $\mathscr{D}$ is a faithful representation of the quotient group $\mathcal{G}/\ker\mathscr{D}$.

19. Representation of the Galilei group

In exercise **11.** you already encountered the Galilei group Gal(3).

i. Show that a 5-dimensional linear representation of Gal(3) consists of the matrices of the type

$$\begin{pmatrix} \mathscr{R} & \vec{v} & \vec{a} \\ \vec{0}^{\mathsf{T}} & 1 & \tau \\ \vec{0}^{\mathsf{T}} & 0 & 1 \end{pmatrix}, \tag{1}$$

where $\vec{0}^{\mathsf{T}}$ denotes a row vector with three zero entries, while $\vec{v}$ and $\vec{a}$ are two column vectors with three arbitrary real entries and $\mathscr{R}$ is a three-dimensional rotation matrix, $\mathscr{R} \in \mathrm{SO}(3)$.

ii. Is this representation faithful? Is it fully reducible? (As you will learn later, the — negative — answer reflects the fact that the Galilei group is not compact.)

20. A matrix representation of a well-known finite group

Consider the $n \times n$ matrix

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{2}$$

i.e. $T_{ij} = 1$ for all $j = i + 1$ and for $(i = m, j = 1)$, and otherwise $T_{ij} = 0$.

i. Construct a group $\tilde{\mathcal{G}}$ from T via matrix multiplication. The matrix group $\tilde{\mathcal{G}}$ is a representation of a finite group $\mathcal{G}$ which you already know; which one?

ii. With the help of Maschke's theorem we know that the representation $\tilde{\mathcal{G}}$ is completely reducible. What are all irreducible representations of $\tilde{\mathcal{G}}$? Calculate for this purpose the eigenspaces of T.

21. Symmetry in a classical many-body system

Consider a planar molecule, in the absence of external forces, with $N > 1$ interacting atoms at coordinates $\mathbf{x}_{i,0} + \mathbf{x}_i \equiv (x_{i,0} + x_i, y_{i,0} + y_i)$ for $i \in \{1, ..., N\}$, where $\mathbf{x}_{i,0} = (x_{i,0}, y_{i,0})$ is the equilibrium position of the i-th atom with mass m_i.

i. Explain why the part of the Hamilton function of the molecule relevant for the description of small oscillations of the atoms about their equilibrium positions may be written as

$$H = \frac{1}{2}\dot{\boldsymbol{X}}^{\mathsf{T}} M^2 \dot{\boldsymbol{X}} + \boldsymbol{X}^{\mathsf{T}} V \boldsymbol{X} \tag{3}$$

with $\boldsymbol{X} \equiv (x_1, y_1, x_2, y_2, ...)$. What is the significance of the matrices M and V?

ii. Show that if the molecule is invariant under the transformations $\boldsymbol{X} \to \boldsymbol{X'} = \mathscr{D}(g)^{-1}\boldsymbol{X}$, where $\mathscr{D}$ is a $2N$-dimensional representation of a group $\mathcal{G}$, then this implies the commutator $[V', \mathscr{D}'] = 0$ where $\mathscr{D}' = M\mathscr{D}M^{-1}$ and $V' = M^{-1}VM^{-1}$.

iii. A normal mode of the system is defined as a solution to the equations of motion in which all atoms oscillate with the same frequency. Show that for a normal mode, $M\boldsymbol{X}$ is an eigenvector of V' and that if $M\boldsymbol{X}$ is an eigenvector, then so is $\mathscr{D}'M\boldsymbol{X}$. Under which conditions is $\mathscr{D}'$ a reducible representation and if so, what do the invariant subspaces correspond to?

iv. Let us consider two examples of the system with identical particles ($m_i = m$ for all $i \in \{1, ..., N\}$), e.g. a benzene ring. The symmetry group $\mathcal{G}$ will be in either case the cyclic group C_N, only the matrix representation will be different. In the first case the generator of the representation, T_1, rotates each particle through the same angle, i.e.

$$\mathbf{x}_i \to \mathbf{x}'_i = \begin{pmatrix} \cos\frac{2\pi}{N} & \sin\frac{2\pi}{N} \\ -\sin\frac{2\pi}{N} & \cos\frac{2\pi}{N} \end{pmatrix} \mathbf{x}_i \tag{4}$$

In the second case, the generator of the representation, T_2, performs a cyclic permutation of the particles, $\mathbf{x}_i \to \mathbf{x}'_i = \mathbf{x}_{i+1}$ for $i \in \{1, ..., N-1\}$ and $\mathbf{x}_N \to \mathbf{x}'_N = \mathbf{x}_1$. What is the relation between these two representations? What does each of the two symmetries tell us about the matrix V? Into which irreducible representations do the representations generated by T_1 and T_2 split?