Tutorial sheet 4

14. Center, centralizer and normalizer

Let $\mathcal{G}$ be a group. The *center* $Z(\mathcal{G})$ of $\mathcal{G}$ is the set of elements $g \in \mathcal{G}$ that commute with every element of $\mathcal{G}$:

$$Z(\mathcal{G}) = \{g \in \mathcal{G} \mid \forall g' \in \mathcal{G}, gg' = g'g\}.$$

If $\mathcal{S}$ denotes a subset of $\mathcal{G}$, the *centralizer* $C_{\mathcal{G}}(\mathcal{S})$ of $\mathcal{S}$ is the set of elements $g \in \mathcal{G}$ that commute with each element of $\mathcal{S}$:

$$C_{\mathcal{G}}(\mathcal{S}) = \{g \in \mathcal{G} \mid \forall g' \in \mathcal{S}, gg' = g'g\}.$$

Eventually, the *normalizer* $N_{\mathcal{G}}(\mathcal{S})$ of $\mathcal{S}$ is defined by

$$N_{\mathcal{G}}(\mathcal{S}) = \{g \in \mathcal{G} \mid g\mathcal{S} = \mathcal{S}g\}.$$

i. Show that $Z(\mathcal{G})$, $C_{\mathcal{G}}(\mathcal{S})$, and $N_{\mathcal{G}}(\mathcal{S})$ are subgroups of $\mathcal{G}$. How are $C_{\mathcal{G}}(\mathcal{S})$ and $N_{\mathcal{G}}(\mathcal{S})$ related when $\mathcal{S}$ consists of a single element?

ii. Show that $Z(\mathcal{G})$ is Abelian and normal in $\mathcal{G}$, and that $C_{\mathcal{G}}(\mathcal{S})$ is normal in $N_{\mathcal{G}}(\mathcal{S})$. Note that the centralizer is not necessarily normal in $\mathcal{G}$, nor Abelian.

iii. Show that if $\mathcal{H}$ is a subgroup of $\mathcal{G}$, then $\mathcal{H}$ is a normal subgroup of its normalizer $N_{\mathcal{G}}(\mathcal{H})$.

15. Pauli matrices

Consider the set consisting of the two-dimensional unit matrix and the Pauli matrices

$$\Sigma = \{\mathbb{1}_2, \sigma_1, \sigma_2, \sigma_3\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

i. Why is the set Σ not a group with respect to matrix multiplication? Extend Σ to a group $\mathcal{G}$ by closing the matrix multiplication.

ii. Calculate the center of $\mathcal{G}$ denoted by $Z(\mathcal{G})$ and identify it with a discrete group you already know.

iii. Determine the quotient group $\mathcal{G}/Z(\mathcal{G})$. Interpret and explain this result (which you may find confusing at first).

iv. Calculate all conjugacy classes of $\mathcal{G}$.

v. Calculate the centralizer of all elements of $\mathcal{G}$.

16. Group homomorphisms

Consider two groups $\mathcal{G}$ and $\mathcal{G}'$ with neutral elements e and e', respectively, and let $f : \mathcal{G} \to \mathcal{G}'$ be a homomorphism. Prove that the following statements hold:

i. $f(e) = e'$ and $f(g^{-1}) = \big(f(g)\big)^{-1}$;

ii. $\operatorname{im} f$ is a subgroup of $\mathcal{G}'$ and $\ker f$ is a normal subgroup of $\mathcal{G}$;

iii. more generally, f sends a subgroup $\mathcal{H}$ of $\mathcal{G}$ to a subgroup of $\mathcal{G}'$, and the "preimage" by f of a subgroup $\mathcal{H}'$ of $\mathcal{G}'$ is a subgroup of $\mathcal{G}$;

iv. f is injective if and only if $\ker f = \{e\}$.

17. Inner automorphisms

Let $\mathcal{G}$ be a group. For every $a \in \mathcal{G}$, one defines a mapping $\phi_a : \mathcal{G} \to \mathcal{G}$ by $g \mapsto \phi_a(g) = aga^{-1}$.

i. Check that ϕ_a is an isomorphism — it is called an *inner automorphism* of $\mathcal{G}$.

ii. Show that the set of inner automorphisms of $\mathcal{G}$ is a subgroup $\mathrm{Inn}(\mathcal{G})$ of the group of automorphisms of $\mathcal{G}$.

iii. Show that ϕ provides a group homomorphism from $\mathcal{G}$ onto $\mathrm{Inn}(\mathcal{G})$. What is the kernel of this homomorphism?