Tutorial sheet 3

12. Conjugacy classes of the three-dimensional rotations

Throughout this exercise, we consider rotations about axes going through a fixed point of three-dimensional space. Let $\mathscr{R}(\vec{n},\alpha)$ denote the rotation through an angle α about the axis with direction $\vec{n}$.

i. Show that for any arbitrary rotation $\mathscr{R}$, the product $\mathscr{R}\mathscr{R}(\vec{n},\alpha)\mathscr{R}^{-1}$ is a rotation through α about the axis with direction $\vec{n}' = \mathscr{R}\vec{n}$.

ii. Deduce from the result of **i.** the conjugacy classes of the group of three-dimensional rotations.

13. A quantum-mechanical problem with D_n symmetry

In the lecture, you saw that the energy-eigenfunctions of a quantum-mechanical problem in one spatial dimension with an even potential $V(x)$ are necessarily either even or odd. The present exercise relies on the same idea.

i. Consider the (x, y)-plane. Let $\mathscr{R}_n$ denote the two-dimensional rotation through an angle $2\pi/n$ (with $n \in \mathbb{N}^*$) around the origin and $\mathscr{S}_y$ denote the reflection across the x-axis.

Show that $\mathscr{R}_n$ and $\mathscr{S}_y$ with the usual composition of geometrical transformations generate a finite group D_n, which for $n \geq 2$ is the symmetry group of an n-sided regular polygon[1] centered on the origin and with one corner on the x-axis. (For $n = 1$, D_1 is the symmetry group of the figure consisting of two points at $x = x_0, y = y_0$ and $x = x_0, y = -y_0$ respectively, with $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^*$.) What is the order of D_n? Check that for every element $\mathscr{R} \in \mathrm{D}_n$, the identity $\mathscr{S}_y^{-1}\mathscr{R}\mathscr{S}_y = \mathscr{R}^{-1}$ holds.

ii. Consider the motion in two spatial dimensions — in the (x, y)-plane — of a particle in a potential $V(x, y)$ with D_n symmetry about the origin. If $|\psi\rangle$ denotes a state vector of the system and $\psi(x, y)$ the corresponding wave function in position representation, then $\hat{\mathscr{R}}|\psi\rangle$ denotes the state vector corresponding to the wave function ψ evaluated at the point $\mathscr{R}(x, y)$ with $\mathscr{R} \in \mathrm{D}_n$.

a) Let $|\psi_E\rangle$ denote an eigenstate of the Hamiltonian $\hat{H}$ of the system. Show that

$$|\psi_k\rangle = \sum_{p=1}^{n} \mathrm{e}^{2\mathrm{i}\pi kp/n} \big(\hat{\mathscr{R}}_n\big)^p |\psi_E\rangle \quad \text{for } k \in \{1, \dots, n\}$$

and

$$|\psi_\pm\rangle = \big(\hat{\mathbb{1}} \pm \hat{\mathscr{S}}_y\big)|\psi_E\rangle$$

are also eigenstates of $\hat{H}$, where $\hat{\mathbb{1}}$ denotes the identity operator on the Hilbert space of the system.

b) Show the orthogonality relations $\langle\psi_k|\psi_{k'}\rangle = 0$ for $k, k' \in \{1, \dots, n\}$ with $k \neq k'$ and $\langle\psi_+|\psi_-\rangle = 0$.

c) Show that $|\psi_k\rangle$ is eigenvector of $\hat{\mathscr{R}}_n$ and $|\psi_\pm\rangle$ eigenvector of $\hat{\mathscr{S}}_y$. What are the respective eigenvalues?

d) Why does a joint eigenbasis of $\hat{\mathscr{R}}_n$, $\hat{\mathscr{S}}_y$, and $\hat{H}$ exist if and only if $n \in \{1, 2\}$? To answer this question calculate the action of $\hat{\mathscr{S}}_y$ on the state vector

$$|\psi_{k,\pm}\rangle = \sum_{p=1}^{n} \mathrm{e}^{2\mathrm{i}\pi kp/n} \big(\hat{\mathscr{R}}_n\big)^p \big(\hat{\mathbb{1}} \pm \hat{\mathscr{S}}_y\big)|\psi_E\rangle \quad \text{with } k \in \{1, \dots, n\}.$$

[1] OK, you may have difficulty picturing in your head the "digon" with $n = 2$. You may replace it with a non-square rectangle with its sides parallel to the coordinate axes, and let the length of the sides parallel to the y-axis go to 0.