Tutorial sheet 2

6. Equivalence relation

Let $\mathcal{G}$ be a group and $\mathcal{H} \subset \mathcal{G}$ a subset of $\mathcal{G}$. Show that the relation $\underset{\text{mod } \mathcal{H}}{\sim}$ defined by

$$g \underset{\text{mod } \mathcal{H}}{\sim} g' \quad \Leftrightarrow \quad g^{-1}g' \in \mathcal{H}$$

for all $g, g' \in \mathcal{G}$ defines an equivalence relation if and only if $\mathcal{H}$ is a subgroup of $\mathcal{G}$.

7. Cosets

Let $\mathcal{G}$ be a group and $\mathcal{H}$ one of its subgroups. Show that the inverses of the elements of a left coset of $\mathcal{H}$ form a right coset of $\mathcal{H}$.

8. Conjugacy classes

Show the following properties:

i. The unit element of a group forms a conjugacy class by itself.

ii. In an Abelian group, each element forms a conjugacy class by itself.

iii. In a group, all the elements of the same conjugacy class are of the same order.

iv. Let g, g' be two elements of a group $\mathcal{G}$ whose unit element is denoted by e. If $(gg')^n = e$ with $n \in \mathbb{N}^*$, then $(g'g)^n = e$.

9. Cosets and conjugacy classes in D_3

In exercise **3.** you already encountered the symmetry group $\mathrm{D}_3 = \{e, c_3, c_3^2, \sigma_1, \sigma_2, \sigma_3\}$ of an equilateral triangle and you found its various subgroups.

i. Determine the right cosets of the subgroup $\mathrm{C}_3 \equiv \{e, c_3, c_3^2\}$, and show that C_3 is normal in D_3.

ii. Consider now the subgroup $\mathrm{C}_2 \equiv \{e, \sigma_1\}$. Is it normal in D_3? Determine all subgroups of D_3 that are conjugate to C_2.

10. Normal subgroups

Show the following results:

i. The intersection $\mathcal{H} \cap \mathcal{K}$ of two normal subgroups $\mathcal{H}, \mathcal{K}$ of a group $\mathcal{G}$ is also a normal subgroup of $\mathcal{G}$.

ii. Any subgroup of index 2 is normal.

iii. The set A_n of even permutations of n elements is a normal subgroup of S_n.

iv. Let $n \in \mathbb{N}^*$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The regular (i.e. invertible) $n \times n$ matrices with entries in $\mathbb{K}$ form a group, denoted by $\mathrm{GL}(n, \mathbb{K})$. The matrices of $\mathrm{GL}(n, \mathbb{K})$ with determinant 1 form a normal subgroup $\mathrm{SL}(n, \mathbb{K}) \lhd \mathrm{GL}(n, \mathbb{K})$.

11. Galilei group

The Galilei group Gal(3) is the set of all possible transformations between the space-time coordinates of two inertial reference frames in non-relativistic physics, with spatial coordinates measured in right-handed orthonormal systems and a constant direction of time ("time arrow"). That is, Gal(3) consists of space and time translations, three-dimensional rotations, and of "proper Galilei transformations" (or boosts) — where the latter correspond to the case of inertial frames in uniform linear motion with respect to each other — and their compositions.

i. To refresh your knowledge from a past Classical Mechanics lecture, write down a (non-trivial!) example of transformation $(t, x, y, z) \to (t', x', y', z')$ for each of the four classes of Galilei transformations listed above.

ii. Check that the Galilei group with the composition of transformations... fulfills the group axioms!

iii. Show that the translations form a normal subgroup of Gal(3).

iv. Give examples showing that neither the subgroup of three-dimensional rotations nor that of Galilei boosts is normal in Gal(3).