# Tutorial sheet 14 ("Mock exam")

## 50. Dihedral group $D_4$

Let $D_4$ denote the dihedral group of order 8, i.e. the symmetry group of a square.

**i.** You might be tempted to consider that $D_4$ can be generated by a rotation $r_4$ through $90^o$ and a reflexion $s$. Instead of that, explain how the 4 elements $\{s, sr_4, sr_4^2, sr_4^3\}$ can be viewed as rotations through $180^o$.

**ii.** The group $D_4$ consists of 5 conjugacy classes. Can you deduce from this information the number of irreducible representations of $D_4$ and their respective dimensions? Explain your result.

**iii.** Assume that the square defines the $(x, y)$-plane and consider a system of coordinates $(x, y, z)$ such that the square vertices sit at $(\pm 1, \pm 1, 0)$. Construct a three-dimensional representation of $D_4$ and compute the corresponding characters. Is this representation reducible?

## 51. Coulomb potential

Consider the motion of a particle in a Coulomb potential, described by the Lagrange function

$$\mathcal{L} = \frac{m}{2}\dot{\vec{r}}^2 + \frac{k}{|\vec{r}|}, \tag{1}$$

where $m$ is the mass of the particle and $k > 0$ some coupling constant.

**i.** What are the corresponding equations of motion?

**ii.** Consider the two kinds of infinitesimal transformations

$$\vec{r} \to \mathcal{T}_1(\vec{r}) = \vec{r} + \delta O\, \vec{r}, \tag{2}$$

where $\delta O$ is an infinitesimal anti-symmetric $3 \times 3$ matrix, $\delta O^{\mathsf{T}} = -\delta O$, and

$$\vec{r} \to \mathcal{T}_2(\vec{r}) = \vec{r} + 2(\delta\vec{a} \cdot \vec{r})\vec{p} - (\delta\vec{a} \cdot \vec{p})\vec{r} - (\vec{r} \cdot \vec{p})\delta\vec{a}, \tag{3}$$

where $\delta\vec{a}$ is an infinitesimal vector of $\mathbb{R}^3$ and $\vec{p}$ the (linear) momentum of the particle.

**a)** Show that both infinitesimal transformations correspond to symmetries of the Lagrange function (1).

*Hint*: Expand $\mathcal{L}$ to first order in $\delta O$ and $\delta\vec{a}$.

**b)** To which symmetry group does transformation (2) correspond?

**iii)** According to Noether's theorem, one can associate conserved quantities to both symmetries. Show that these quantities are

$$\vec{L} = \vec{r} \times \vec{p} \quad , \quad \vec{A} = \vec{p} \times (\vec{r} \times \vec{p}) - \frac{mkr}{|\vec{r}|}, \tag{4}$$

respectively. Explain their physical significance.

## 52. Heisenberg group

Let $\hat{x}$, $\hat{p}$ denote the position and momentum operators in the $x$-direction and $\hat{\mathbb{1}}$ be the identity operator.

**i.** Show that the three operators $\{\hat{x}, \hat{p}, \hat{\mathbb{1}}\}$ form a Lie algebra (the Heisenberg algebra), which is closed under taking commutators. Give its commutation relations and the structure constants.

**ii.** Consider the three-dimensional upper triangular matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with } a, b, c \in \mathbb{R}. \tag{5}$$

Show that the matrices of this type constitute a Lie group, often referred to as the *Heisenberg group*. Of which form are the matrices of the corresponding Lie algebra? Show that the latter is isomorphic to that considered in **i.**.

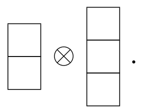
**iii.** Is the Heisenberg group compact?

## 53. Irreducible representations of SU(5)

**i.** Considering as usual that $\square$ stands for the defining (**5**) representation of SU(5), which Young diagrams are associated to the conjugate representation $\bar{\mathbf{5}}$ and to the adjoint representation? What is the dimension of the latter?

**ii.** Which (simple) Young diagram corresponds to the **10** representation? Inspiring yourself from the "correspondence" between the **5** and the $\bar{\mathbf{5}}$, can you then guess which Young diagram stands for the $\bar{\mathbf{10}}$ representation?

**iii.** Compute the outer product of the Young diagrams consisting of single columns with two resp. three boxes:



Viewing these diagrams as irreducible representations of SU(5), compute the dimensions of all irreducible representations involved and write down explicitly the corresponding Clebsch–Gordan series in the form $\mathbf{?} \otimes \mathbf{??} = \cdots$.

The group SU(5) was proposed as gauge group of a "grand unified theory" (GUT) encompassing the strong and electroweak interactions of particle physics (cf. Georgi–Glashow model).