Tutorial sheet 13

46. Theories for a classical vector field

i. Consider a (4-)vector field with components $A^\mu(\mathsf{x})$, described by the Lagrange density

$$\mathscr{L}_0 = -\frac{1}{2}(\partial^\mu A^\nu)(\partial_\mu A_\nu - \partial_\nu A_\mu), \tag{1}$$

where the 4 components A^ν are to be considered as independent fields.

a) Write down the corresponding Euler–Lagrange equations. For the sake of brevity, you may introduce the notation $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$.

b) The Lagrange density (1) is invariant under the transformations $A^\mu \to A'^\mu = A^\mu + \varepsilon^\mu$ where the ε^μ are the components of an arbitrary, time- and space-independent 4-vector. Derive the Noether current J^μ associated with this symmetry. What is the conservation equation $\partial_\mu J^\mu = 0$ equivalent to?

ii. Consider now a vector field with the Lagrange density

$$\mathscr{L}_{\text{st.}} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{2}$$

and write down the corresponding Euler–Lagrange equations. What do you notice?

iii. How do the Lagrange densities (1) and (2) behave under the transformation $A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi$, with χ an arbitrary (differentiable) function of the space-time coordinates? Is the group of transformations of that type a symmetry of the corresponding physical systems?

47. Isospin symmetry

i. Consider a theory for two Dirac spinor-fields ψ_p, ψ_n (standing respectively for a proton and a neutron) with the Lagrange density

$$\mathscr{L}_N = \bar{\psi}_p\big(\mathrm{i}\gamma^\mu\partial_\mu - m_p\big)\psi_p + \bar{\psi}_n\big(\mathrm{i}\gamma^\mu\partial_\mu - m_n\big)\psi_n, \tag{3}$$

where the γ^μ, $\mu = 0, 1, 2, 3$ are Dirac matrices, while the adjoint spinors $\bar{\psi}_p$, $\bar{\psi}_n$ are to be treated as fields independent of ψ_p, ψ_n.

a) For the sake of completeness, derive the equations of motions obeyed by ψ_p and ψ_n. Are they known to you?

b) If $m_p \neq m_n$, the Lagrange density (3) is invariant under a "global" $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry whose transformations are of the form

$$\begin{cases} \psi_p \to \psi'_p = \mathrm{e}^{-\mathrm{i}\Lambda_p}\,\psi_p \quad , \quad \bar{\psi}_p \to \bar{\psi}'_p = \mathrm{e}^{\mathrm{i}\Lambda_p}\,\bar{\psi}_p \\ \psi_n \to \psi'_n = \mathrm{e}^{-\mathrm{i}\Lambda_n}\,\psi_n \quad , \quad \bar{\psi}_n \to \bar{\psi}'_n = \mathrm{e}^{\mathrm{i}\Lambda_n}\,\bar{\psi}_n \end{cases} \tag{4}$$

where Λ_p and Λ_n are time- and space-independent real constants. Derive the corresponding Noether currents. Can you guess what the associated conserved charges are?

ii. Assume now that $m_p = m_n$.

a) Introducing the "two-dimensional vector" with Dirac-spinor entries

$$\Psi \equiv \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$$

and its adjoint $\bar{\Psi} \equiv \big(\bar{\psi}_p \;\; \bar{\psi}_n\big)$, show that one can recast the Lagrange density (3) in the form $\mathscr{L}_N = \bar{\Psi} D \Psi$

where D is a "2×2" matrix proportional to $\mathbb{1}_2$, and that it is now invariant under transformations $\Psi \to \Psi' = \mathcal{U}\Psi$, $\bar{\Psi} \to \bar{\Psi}' = \bar{\Psi}\mathcal{U}^\dagger$ with a time- and space-independent $\mathcal{U} \in \mathrm{U}(2)$.

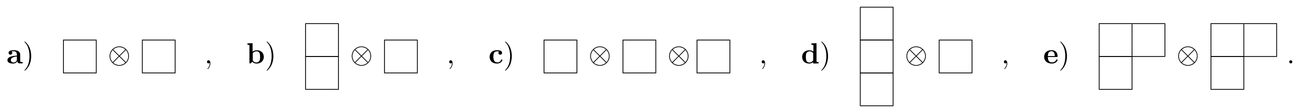
b) Writing $\mathcal{U}$ as $e^{i\theta}U$ with $e^{i\theta} \in \mathrm{U}(1)$ and $U \in \mathrm{SU}(2)$, write down infinitesimal transformations of the fields Ψ and $\bar{\Psi}$ (*hint*: write the SU(2) matrix in exponential form) and derive the Noether currents associated with the U(1) and SU(2) symmetries. SU(2) is the so-called "isospin symmetry" of the system (cf. exercise **45**).

If you have some notions of particle physics, can you guess what the conserved charge associated with the U(1) part corresponds to?

This exercise will be continued in exercise **49.**.

48. Young diagrams and representations of SU(N)

i. Compute the following products of Young diagrams:



ii. Viewing  as the defining ($\boldsymbol{N}$) representation of SU(N), compute the dimensions of all irreducible representations involved in question **i.** for $N = 2, 3, 4$ and write down explicitly the corresponding Clebsch–Gordan series in the form $\boldsymbol{N} \otimes \boldsymbol{N} = \cdots$, and so on.
(For $N = 2$, you should be able to double-check the results with your knowledge from the "SU(2)-SO(3)" lecture; for $N = 3$, those with some knowledge of the quark model may recognize part of the results.)

49. Isospin symmetry (2)

i. Consider now a theory with 3 real scalar fields π_1, π_2, π_3, collectively denoted by a 3-vector $\vec{\pi}$, described by the Lagrange density

$$\mathscr{L}_\pi = -\frac{1}{2}\big(\partial_\mu\vec{\pi}\big)\cdot\big(\partial^\mu\vec{\pi}\big) - \frac{1}{2}m_\pi^2\,\vec{\pi}^2. \tag{5}$$

Check that this Lagrange density is invariant under transformations $\vec{\pi} \to \vec{\pi}' = O\vec{\pi}$ with a time- and space-independent $O \in \mathrm{SO}(3)$ and derive the corresponding Noether currents. Why is the present invariance possibly the "same" isospin symmetry as in exercise **47.ii.**?

ii. The fields Ψ introduced in **47.ii.a)** and $\vec{\pi}$ are coupled together by a term

$$\mathscr{L}_{N-\pi} = ig\big(\bar{\Psi}\gamma^5\vec{\sigma}\Psi\big)\cdot\vec{\pi} \tag{6}$$

where $\vec{\sigma}$ denotes as usual a 3-vector whose entries are the Pauli matrices, while γ^5 is a matrix that acts on the spinorial degrees of freedom of the fields ψ_p, ψ_n and plays no role in the following.

a) Compute the behavior of the term $\bar{\Psi}\gamma^5\vec{\sigma}\Psi$ under infinitesimal isospin transformations (*hint*: you may need the identity $[\sigma_i, \alpha_j\sigma_j] = 2i\alpha_j\sum_k \epsilon_{ijk}\sigma_k$) and deduce therefrom the invariance of the interaction Lagrange density (6) under isospin.

b) Derive the Noether current(s) associated with isospin symmetry for the theory with Lagrange density

$$\mathscr{L}_{N+\pi} = \mathscr{L}_N + \mathscr{L}_\pi + \mathscr{L}_{N-\pi}. \tag{7}$$

c) The particle-physics fans may rewrite the interaction term (6) in terms of the proton and neutron fields ψ_p, ψ_n and of the physical pions (π^+, π^-, π^0) given by

$$\pi_1 = \frac{1}{\sqrt{2}}(\pi^+ + \pi^-) \quad , \quad \pi_2 = -\frac{i}{\sqrt{2}}(\pi^+ - \pi^-) \quad , \quad \pi_3 = \pi^0.$$