Tutorial sheet 12

43. SO(3) matrices in the spherical basis

Consider the spin-1 irreducible representation of SU(2).

i. Write down the generators $\mathscr{J}_1^{(3)}, \mathscr{J}_2^{(3)}, \mathscr{J}_3^{(3)}$ in the "spherical" basis $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$ consisting of the eigenvectors of $\mathscr{J}_3^{(3)}$ (*hint*: write down first the matrices $\mathscr{J}_+^{(3)}, \mathscr{J}_-^{(3)}$).

ii. Show that Wigner's small d-matrix reads

$$d^1(\psi) \equiv \mathrm{e}^{-\mathrm{i}\psi\mathscr{J}_2^{(3)}} = \begin{pmatrix} \frac{1+\cos\psi}{2} & -\frac{\sin\psi}{\sqrt{2}} & \frac{1-\cos\psi}{2} \\ \frac{\sin\psi}{\sqrt{2}} & \cos\psi & -\frac{\sin\psi}{\sqrt{2}} \\ \frac{1-\cos\psi}{2} & \frac{\sin\psi}{\sqrt{2}} & \frac{1+\cos\psi}{2} \end{pmatrix}. \tag{1}$$

44. Standard components of a vector operator

The so-called *standard components* of a vector operator $\hat{\vec{V}}$ are defined as

$$\hat{V}_1^{(1)} \equiv -\frac{1}{\sqrt{2}}(\hat{V}_x + \mathrm{i}\hat{V}_y) \quad , \quad \hat{V}_0^{(1)} \equiv \hat{V}_z \quad , \quad \hat{V}_{-1}^{(1)} \equiv \frac{1}{\sqrt{2}}(\hat{V}_x - \mathrm{i}\hat{V}_y), \tag{2}$$

where $\hat{V}_x, \hat{V}_y, \hat{V}_z$ are the Cartesian components of $\hat{\vec{V}}$.

Starting from the standard components $\hat{V}_m^{(1)}$, $\hat{W}_{m'}^{(1)}$ of two vector operators $\hat{\vec{V}}$, $\hat{\vec{W}}$, one defines operators

$$\big(\hat{\vec{V}} \otimes \hat{\vec{W}}\big)_M^{(J)} = \sum_{m,m'} C_{1,1;m,m'}^{J,M} V_m^{(1)} W_{m'}^{(1)} \tag{3}$$

where the numbers $C_{1,1;m,m'}^{J,M} \equiv \langle 1,1;m,m'\,|\,J,M\rangle$ are the Clebsch–Gordan coefficients relevant to the addition of two spins 1.[1]

i. Show that $\big(\hat{\vec{V}} \otimes \hat{\vec{W}}\big)_0^{(0)}$ is proportional to the scalar product $\hat{\vec{V}} \cdot \hat{\vec{W}}$ of the vector operators.

ii. Show that the three operators $\big(\hat{\vec{V}} \otimes \hat{\vec{W}}\big)_M^{(1)}$ are proportional to the three standard components of the vector operator $\hat{\vec{V}} \times \hat{\vec{W}}$.

iii. Express the five operators $\big(\hat{\vec{V}} \otimes \hat{\vec{W}}\big)_M^{(2)}$ in terms of $\hat{V}_z$, $\hat{V}_\pm \equiv \hat{V}_x \pm \mathrm{i}\hat{V}_y$, $\hat{W}_z$, and $\hat{W}_\pm \equiv \hat{W}_x \pm \mathrm{i}\hat{W}_y$.

45. Isospin and reaction rates

"Isospin" is an approximate SU(2) symmetry of the strong interaction: the proton and the neutron form a doublet (**2**-irrep., isospin $\frac{1}{2}$) where the proton (p) is the state $|j,m\rangle = |\frac{1}{2},\frac{1}{2}\rangle$ and the neutron (n) the state $|\frac{1}{2},-\frac{1}{2}\rangle$. When they bind together, a proton and a neutron form a deuteron (d), with isospin 0 (an "iso-scalar").

In proton-deuteron scattering, one observes (among others) the following two channels:

$$p + d \to \pi^0 + {}^3\mathrm{He} \quad , \quad p + d \to \pi^+ + {}^3\mathrm{H}. \tag{4}$$

Both pions in these reactions are elements of an "iso-triplet" (**3**-irrep., isospin 1) with respectively $\pi^+ = |1,1\rangle$ and $\pi^0 = |1,0\rangle$.[2] In turn, the ^{3}He and ^{3}H nuclei form an iso-doublet: ${}^3\mathrm{He} = |\frac{1}{2},\frac{1}{2}\rangle$, ${}^3\mathrm{H} = |\frac{1}{2},-\frac{1}{2}\rangle$.

[1] See e.g. `http://www-pdg.lbl.gov/2017/reviews/rpp2017-rev-clebsch-gordan-coefs.pdf` for a table with these coefficients.

[2] The $|1,-1\rangle$-state is the negatively charged π^-, which plays no role in this exercise

i. Using the fact that the relevant Hamilton operator is an iso-scalar — which means that isospin is a symmetry of the system —, express the transition amplitudes from the initial to the final states of reactions (4) as the product of a process-dependent coefficient controlled by the symmetry group and a matrix element which you cannot explicitly compute.

ii. Compute the ratio of the transition probabilities $\sigma(p + d \to \pi^0 + {}^3\mathrm{He})$ and $\sigma(p + d \to \pi^+ + {}^3\mathrm{H})$.