Tutorial sheet 11

39. Lie bracket of generators of $\mathfrak{so}(n)$

The Lie algebra $\mathfrak{so}(n)$ of the special orthogonal group $\mathrm{SO}(n)$ is the space of traceless antisymmetric $n \times n$ matrices, where $n \geq 2$. (Troughout this exercise, the position of the indices is irrelevant.)

i. Check that a basis of generators (in the physicists' convention) consists of matrices T^{ab}, with $1 \leq a < b \leq n$, whose ij-entry is $(T^{ab})_{ij} = -\mathrm{i}(\delta^a_i\delta^b_j - \delta^a_j\delta^b_i)$, where δ^k_l is the usual Kronecker symbol. Show that the Lie bracket of two such generators is

$$[T^{ab}, T^{cd}] = -\mathrm{i}\big(\delta^{bc}T^{ad} + \delta^{ad}T^{bc} - \delta^{bd}T^{ac} - \delta^{ac}T^{bd}\big). \tag{1}$$

ii. Show that in the case $n = 3$ you recover results from the lecture.

iii. Denote X a vector with n real components x^j. Given a matrix $O \in \mathrm{SO}(n)$ close to the identity matrix $\mathbb{1}_n$, the transformation $X \to OX \equiv X'$ yields $X' = X + \delta X$ where the components δx^j of δX are "small". Show that one may write

$$\delta x^j = -\frac{\mathrm{i}}{2}\omega_{ab}T^{ab}x^j \quad \text{with} \quad T^{ab} \equiv -\mathrm{i}\left(x^a\frac{\partial}{\partial x^b} - x^b\frac{\partial}{\partial x^a}\right) \tag{2}$$

and with ω_{ab} antisymmetric (and traceless). Convince yourself that the differential operators T^{ab} defined in Eq. (2) are in one-to-one correspondence with the matrices T^{ab} of question **i.**

40. SU(1,1) and SL$(2,\mathbb{R})$

i. The group SU(1,1) consists of the complex 2×2 matrices U with determinant 1 such that $U^\dagger \eta U = \eta$, where $\eta \equiv \mathrm{diag}(1,-1)$.

a) What is the dimension of SU(1,1)?

b) Which equation does an element X of the Lie algebra $\mathfrak{su}(1,1)$ obey? What does that equation imply for the matrix elements of X? Prove that one may write a basis of $\mathfrak{su}(1,1)$ in terms of the Pauli matrices and compute their commutation relations.

c) Is $\mathfrak{su}(1,1)$ isomorphic[1] to the algebra $\mathfrak{so}(3)$?

ii. Consider now the special linear group $\mathrm{SL}(2,\mathbb{R})$.

a) Recall its definition and give its dimension. How is its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ defined? Give a basis in terms of Pauli matrices.

b) Show that the two Lie algebras $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2,\mathbb{R})$ are isomorphic.

41. Two-dimensional representation of U(1)

Consider the map $\mathscr{D}^{(m)}$ from U(1) into SO(2) defined by

$$\mathscr{D}^{(m)}(\mathrm{e}^{\mathrm{i}\alpha}) = \begin{pmatrix} \cos(m\alpha) & \sin(m\alpha) \\ -\sin(m\alpha) & \cos(m\alpha) \end{pmatrix}. \tag{3}$$

For which values of m is $\mathscr{D}^{(m)}$ a representation of the group U(1)? With which physical quantity would you associate the number m?

[1] A homomorphism of Lie algebras is a bijective linear application between the underlying vector spaces that preserves the Lie brackets.

42. A property of $\mathfrak{so}(4)$

In exercise **39.** you saw a family of generators of $\mathfrak{so}(n)$ and their Lie brackets. Take now $n = 4$ and define (beware the signs!)

$$A_1 \equiv \frac{1}{2}(T^{12} - T^{34}) \quad , \quad A_2 \equiv \frac{1}{2}(T^{13} + T^{24}) \quad , \quad A_3 \equiv \frac{1}{2}(T^{14} - T^{23}) \tag{4}$$

and

$$B_1 \equiv \frac{1}{2}(T^{12} + T^{34}) \quad , \quad B_2 \equiv \frac{1}{2}(-T^{13} + T^{24}) \quad , \quad B_3 \equiv \frac{1}{2}(T^{14} + T^{23}). \tag{5}$$

Compute the commutators $[A_i, A_j]$, $[B_i, B_j]$, and $[A_i, B_j]$. How would you be tempted to interpret your findings? (You actually know a physics problem with a "hidden" SO(4) symmetry, which will be the topic of a later exercise.)

We wish you a merry Christmas and a happy new year!