Tutorial sheet 10

36. Lie algebras of classical groups

i. Let $\mathcal{G}$ be a matrix Lie group and $\mathfrak{g}$ the space of matrices such that $M \in \mathfrak{g}$ implies $\mathrm{e}^M \in \mathcal{G}$.

Show that **a)** $\mathfrak{gl}(n)$, **b)** $\mathfrak{sl}(n)$, **c)** $\mathfrak{o}(n) = \mathfrak{so}(n)$, **d)** $\mathfrak{u}(n)$, **e)** $\mathfrak{su}(n)$, respectively associated to the groups GL(n), SL(n), O(n) or SO(n), U(n), SU(n), consist of **a)** arbitrary $n \times n$ matrices, **b)** traceless matrices, **c)** antisymmetric matrices, **d)** antihermitian matrices, **e)** traceless antihermitian matrices.

ii. For each of the above cases, check that the characteristic property (tracelessness, antisymmetry...) of the matrices of $\mathfrak{g}$ is preserved by the commutator, thereby ensuring that $\mathfrak{g}$ is a Lie algebra.

37. An example of the non-surjectivity of the exponential map

Consider the non-compact group SL$(2,\mathbb{R})$: its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ consists of traceless 2×2-matrices with real entries.

i. For any $M \in \mathfrak{sl}(2,\mathbb{R})$, show that $\mathrm{Tr}\, M^{2n} = 2(-\det M)^n$ and $\mathrm{Tr}\, M^{2n+1} = 0$ for all $n \in \mathbb{N}$.

Hint: Consider the characteristic polynomial of M.

ii. Deduce from **i.** that $\mathrm{Tr}\, \mathrm{e}^M \geq -2$ for all $M \in \mathfrak{sl}(2,\mathbb{R})$. Give an example of element of SL$(2,\mathbb{R})$ with a trace strictly smaller than -2 and conclude.

38. Relation between SO(3) and SU(2)

i. The group SO(3) consists of the real 3×3-matrices with determinant 1 that preserve the usual (Euclidean) scalar product, i.e. it describes rotations in $\mathbb{R}^3$. Let $\mathscr{R}_{\vec{n}}(\alpha)$ denote the rotation through the angle α about the axis with unit vector $\vec{n}$.

a) As a warm up, convince yourself that the rotations $\mathscr{R}_{\vec{n}}(\alpha)$ and $\mathscr{R}_{-\vec{n}}(-\alpha)$ are identical and derive the Rodrigues formula

$$\mathscr{R}_{\vec{n}}(\alpha)\vec{x} = (\cos\alpha)\vec{x} + (1-\cos\alpha)(\vec{n}\cdot\vec{x})\vec{n} + (\sin\alpha)\vec{n}\times\vec{x} \qquad \forall \vec{x} \in \mathbb{R}^3. \tag{1}$$

Hint: Decompose $\vec{x}$ in two components parallel and perpendicular to $\vec{n}$.

b) Associate to the rotation $\mathscr{R}_{\vec{n}}(\alpha)$ the vector $\boldsymbol{u}$ with 4 components $\boldsymbol{u} \equiv (\vec{u} \equiv \vec{n}\cos\frac{\alpha}{2}, u_4 \equiv \sin\frac{\alpha}{2})$. Show that $\boldsymbol{u}$ belongs to the unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4$, i.e. that the Euclidean norm $[\vec{u}^2 + (u_4)^2]^{1/2}$ of $\boldsymbol{u}$ equals 1. What happens to $\mathscr{R}_{\vec{n}}(\alpha)$ and to $\boldsymbol{u}$ when you add to α an odd multiple of 2π?

ii. The group SU(2) consists of the unitary 2×2-matrices with determinant 1.

a) (Reminder of your Quantum Mechanics lecture) Check that for every point $\boldsymbol{u} \in \mathbb{S}^3$, the matrix

$$U = u_4 \mathbb{1}_2 - \mathrm{i}\,\vec{u}\cdot\vec{\sigma} \tag{2}$$

is in SU(2), where $\mathbb{1}_2$ is the unit 2×2-matrix while $\vec{\sigma}$ is a “vector” whose entries are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Conversely, show that every matrix of SU(2) can be written in the form (2). Deduce therefrom that SU(2) and $\mathbb{S}^3$ are in bijection to each other. By “exporting” the group law of SU(2) to $\mathbb{S}^3$, one defines a group structure on the latter, which is then isomorphic to SU(2).

b) Let $\vec{n}$ be a unit vector of $\mathbb{R}^3$ (one may write that $\vec{n}$ belongs to the sphere $\mathbb{S}^2$) and $\alpha \in [0, 2\pi]$. Show the identity

$$U_{\vec{n}}(\alpha) \equiv \mathrm{e}^{-\mathrm{i}\alpha\vec{n}\cdot\vec{\sigma}/2} = \left(\cos\frac{\alpha}{2}\right)\mathbb{1}_2 - \mathrm{i}\left(\sin\frac{\alpha}{2}\right)\vec{n}\cdot\vec{\sigma}, \tag{4}$$

which defines the matrix $U_{\vec{n}}(\alpha)$.

iii. Homomorphism between SU(2) and SO(3)

a) To each vector $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ one associates a traceless Hermitian matrix X according to

$$X \equiv \vec{x}\cdot\vec{\sigma} = \begin{pmatrix} x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & -x_3 \end{pmatrix}. \tag{5}$$

Show that the correspondence is bijective, i.e. that to any traceless Hermitian matrix X one can associate a unique vector $\vec{x} \in \mathbb{R}^3$. (*hint*: trace!)

b) Given a unitary matrix $U \in \mathrm{SU}(2)$, one maps the matrix X of Eq. (5) to the matrix $X' = UXU^\dagger$. This defines a linear transformation $\vec{x} \mapsto \vec{x}\,' = \mathcal{T}(U)\vec{x}$ such that $X' = \vec{\sigma}\cdot\vec{x}\,'$. Show that $\mathcal{T}(U)$ is a rotation.

Hint: You may either show that $\mathcal{T}(U)$ is an isometry — preserving $|\vec{x}|$ — with determinant 1, or compute X' by introducing an explicit form for U.

c) One can even show — if you did not use that result in **b)** — that $\mathcal{T}(U_{\vec{n}}(\alpha))$ is the rotation $\mathscr{R}_{\vec{n}}(\alpha)$. Show that $\mathcal{T}$ is a group homomorphism from SU(2) onto SO(3). Is it an isomorphism? If no, what is its kernel?

d) Summarize for yourself what you have learned of the relationships between SU(2), $\mathbb{S}^3$ and SO(3).