Tutorial sheet 1

1. On the last two group axioms

Let $(\mathcal{G}, \cdot)$ be a group.

i. Show that the left neutral element e of $\mathcal{G}$, defined by $eg = g$ for all $g \in \mathcal{G}$, is also a right neutral element of $\mathcal{G}$, i.e. $ge = g$ for all $g \in \mathcal{G}$, and that there is exactly one such neutral element in $\mathcal{G}$.

ii. Show that the left inverse of an element $g \in \mathcal{G}$, defined by $g^{-1}g = e$, is also a right inverse of g, and that every element $g \in \mathcal{G}$ has exactly one inverse.

iii. What is the inverse of $g \cdot h$ for $g, h \in \mathcal{G}$?

iv. Show that each row and column of the multiplication table (Cayley table) of a finite group contains every element once.

2. Subgroups

i. Give a proof of the following theorem: A non-empty subset $\mathcal{H} \subset \mathcal{G}$ is a subgroup of the group $\mathcal{G}$ if and only if the following condition holds: $g, h \in \mathcal{H} \Longrightarrow g \cdot h^{-1} \in \mathcal{H}$.

ii. Show that for a *finite* group $\mathcal{G}$, the following condition is sufficient: $g, h \in \mathcal{H} \Longrightarrow g \cdot h \in \mathcal{H}$. Show with a simple example that this condition is not sufficient for an infinite group.

iii. Let $\mathcal{H}, \mathcal{K}$ be two subgroups of the group $\mathcal{G}$. Show that their intersection $\mathcal{H} \cap \mathcal{K}$ is also a subgroup of $\mathcal{G}$.

3. Dihedral group of order 6

Let $\mathrm{D}_3 = \{e, c_3, c_3^2, \sigma_1, \sigma_2, \sigma_3\}$ be the symmetry group of an equilateral triangle with corners A_1, A_2, A_3 in the Euclidean plane, where c_3 is the rotation by 120° around the center O of the triangle, and σ_i is the reflection across the axis through O and A_i.

i. Check that D_3 can also be identified with the group of all permutations of the corners of the triangle — that is, D_3 is isomorphic to the symmetric group S_3.

ii. Determine the multiplication table (Cayley table) of D_3.

iii. Give the list of all subgroups of D_3.

4. Subgroup generated by a permutation

Let $\sigma \in \mathrm{S}_6$ be the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 1 & 5 & 2 \end{pmatrix} = (134)(26),$$

and let $\mathcal{G} \subset \mathrm{S}_6$ be the smallest subgroup of S_6 containing σ. How many elements does $\mathcal{G}$ have? Compute, for this purpose, the powers σ^2, σ^3, σ^4,... of σ in cycle notation. What is σ^{-1}?

5. Matrix representation of the symmetric group S_n

Let $\sigma \in \mathrm{S}_n$ be a permutation of the n numbers $(1, \ldots, n)$ and let $D(\sigma)$ be the square $n \times n$-matrix whose i, j entry $D(\sigma)_{ij}$ equals 1 if $i = \sigma(j)$ and 0 otherwise for $i, j \in \{1, \ldots, n\}$.

i. Write down the six 3×3 matrices representing the elements of the symmetry group S_3 and check on a few of them[1] that the property $D(\sigma)D(\sigma') = D(\sigma\sigma')$ holds.

ii. Coming back to the case of an arbitrary $n \in \mathbb{N}^*$, interpret the trace and the determinant of the matrix $D(\sigma)$ representing a permutation $\sigma \in S_n$.

[1] Do not check the 25 possible non-trivial matrix products! Unless you automatize the procedure with some symbolic-algebra program.