Tutorial sheet 1

1. On the last two group axioms

Let $(\mathcal{G}, \cdot\,)$ be a group.

i. Show that the left neutral element e of $\mathcal{G}$, defined by $eg = g$ for all $g \in \mathcal{G}$, is also a right neutral element of $\mathcal{G}$, i.e. $ge = g$ for all $g \in \mathcal{G}$, and that there is exactly one such neutral element in $\mathcal{G}$.

ii. Show that the left inverse of an element $g \in \mathcal{G}$, defined by $g^{-1}g = e$, is also a right inverse of g, and that every element $g \in \mathcal{G}$ has exactly one inverse.

iii. What is the inverse of $g \cdot h$ for $g, h \in \mathcal{G}$?

iv. Show that each row and column of the multiplication table (Cayley table) of a finite group contains every element once.

2. Subgroups

i. Give a proof of the following theorem: A non-empty subset $\mathcal{H} \subset \mathcal{G}$ is a subgroup of the group $\mathcal{G}$ if and only if the following condition holds: $g, h \in \mathcal{H} \Longrightarrow g \cdot h^{-1} \in \mathcal{H}$.

ii. Show that for a *finite* group $\mathcal{G}$, the following condition is sufficient: $g, h \in \mathcal{H} \Longrightarrow g \cdot h \in \mathcal{H}$. Show with a simple example that this condition is not sufficient for an infinite group.

iii. Let $\mathcal{H}, \mathcal{K}$ be two subgroups of the group $\mathcal{G}$. Show that their intersection $\mathcal{H} \cap \mathcal{K}$ is also a subgroup of $\mathcal{G}$.

3. Dihedral group of order 6

Let $\mathrm{D}_3 = \{e, c_3, c_3^2, \sigma_1, \sigma_2, \sigma_3\}$ be the symmetry group of an equilateral triangle with corners A_1, A_2, A_3 in the Euclidean plane, where c_3 is the rotation by 120° around the center O of the triangle, and σ_i is the reflection across the axis through O and A_i.

i. Check that D_3 can also be identified with the group of all permutations of the corners of the triangle — that is, D_3 is isomorphic to the symmetric group S_3.

ii. Determine the multiplication table (Cayley table) of D_3.

iii. Give the list of all subgroups of D_3.

4. Subgroup generated by a permutation

Let $\sigma \in \mathrm{S}_6$ be the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 1 & 5 & 2 \end{pmatrix} = (134)(26),$$

and let $\mathcal{G} \subset \mathrm{S}_6$ be the smallest subgroup of S_6 containing σ. How many elements does $\mathcal{G}$ have? Compute, for this purpose, the powers σ^2, σ^3, σ^4,... of σ in cycle notation. What is σ^{-1}?

5. Matrix representation of the symmetric group S_n

Let $\sigma \in \mathrm{S}_n$ be a permutation of the n numbers $(1, \ldots, n)$ and let $D(\sigma)$ be the square $n \times n$-matrix whose i, j entry $D(\sigma)_{ij}$ equals 1 if $i = \sigma(j)$ and 0 otherwise for $i, j \in \{1, \ldots, n\}$.

i. Write down the six 3×3 matrices representing the elements of the symmetry group S_3 and check on a few of them[1] that the property $D(\sigma)D(\sigma') = D(\sigma\sigma')$ holds.

ii. Coming back to the case of an arbitrary $n \in \mathbb{N}^*$, interpret the trace and the determinant of the matrix $D(\sigma)$ representing a permutation $\sigma \in \mathrm{S}_n$.

[1]Do not check the 25 possible non-trivial matrix products! Unless you automatize the procedure with some symbolic-algebra program.

Tutorial sheet 2

6. Equivalence relation

Let $\mathcal{G}$ be a group and $\mathcal{H} \subset \mathcal{G}$ a subset of $\mathcal{G}$. Show that the relation $\underset{\text{mod } \mathcal{H}}{\sim}$ defined by

$$g \underset{\text{mod } \mathcal{H}}{\sim} g' \quad \Leftrightarrow \quad g^{-1}g' \in \mathcal{H}$$

for all $g, g' \in \mathcal{G}$ defines an equivalence relation if and only if $\mathcal{H}$ is a subgroup of $\mathcal{G}$.

7. Cosets

Let $\mathcal{G}$ be a group and $\mathcal{H}$ one of its subgroups. Show that the inverses of the elements of a left coset of $\mathcal{H}$ form a right coset of $\mathcal{H}$.

8. Conjugacy classes

Show the following properties:

i. The unit element of a group forms a conjugacy class by itself.

ii. In an Abelian group, each element forms a conjugacy class by itself.

iii. In a group, all the elements of the same conjugacy class are of the same order.

iv. Let g, g' be two elements of a group $\mathcal{G}$ whose unit element is denoted by e. If $(gg')^n = e$ with $n \in \mathbb{N}^*$, then $(g'g)^n = e$.

9. Cosets and conjugacy classes in D_3

In exercise **3.** you already encountered the symmetry group $\mathrm{D}_3 = \{e, c_3, c_3^2, \sigma_1, \sigma_2, \sigma_3\}$ of an equilateral triangle and you found its various subgroups.

i. Determine the right cosets of the subgroup $\mathrm{C}_3 \equiv \{e, c_3, c_3^2\}$, and show that C_3 is normal in D_3.

ii. Consider now the subgroup $\mathrm{C}_2 \equiv \{e, \sigma_1\}$. Is it normal in D_3? Determine all subgroups of D_3 that are conjugate to C_2.

10. Normal subgroups

Show the following results:

i. The intersection $\mathcal{H} \cap \mathcal{K}$ of two normal subgroups $\mathcal{H}, \mathcal{K}$ of a group $\mathcal{G}$ is also a normal subgroup of $\mathcal{G}$.

ii. Any subgroup of index 2 is normal.

iii. The set A_n of even permutations of n elements is a normal subgroup of S_n.

iv. Let $n \in \mathbb{N}^*$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The regular (i.e. invertible) $n \times n$ matrices with entries in $\mathbb{K}$ form a group, denoted by $\mathrm{GL}(n, \mathbb{K})$. The matrices of $\mathrm{GL}(n, \mathbb{K})$ with determinant 1 form a normal subgroup $\mathrm{SL}(n, \mathbb{K}) \triangleleft \mathrm{GL}(n, \mathbb{K})$.

11. Galilei group

The Galilei group Gal(3) is the set of all possible transformations between the space-time coordinates of two inertial reference frames in non-relativistic physics, with spatial coordinates measured in right-handed orthonormal systems and a constant direction of time ("time arrow"). That is, Gal(3) consists of space and time translations, three-dimensional rotations, and of "proper Galilei transformations" (or boosts) — where the latter correspond to the case of inertial frames in uniform linear motion with respect to each other — and their compositions.

i. To refresh your knowledge from a past Classical Mechanics lecture, write down a (non-trivial!) example of transformation $(t, x, y, z) \to (t', x', y', z')$ for each of the four classes of Galilei transformations listed above.

ii. Check that the Galilei group with the composition of transformations… fulfills the group axioms!

iii. Show that the translations form a normal subgroup of Gal(3).

iv. Give examples showing that neither the subgroup of three-dimensional rotations nor that of Galilei boosts is normal in Gal(3).

Tutorial sheet 3

12. Conjugacy classes of the three-dimensional rotations

Throughout this exercise, we consider rotations about axes going through a fixed point of three-dimensional space. Let $\mathscr{R}(\vec{n},\alpha)$ denote the rotation through an angle α about the axis with direction $\vec{n}$.

i. Show that for any arbitrary rotation $\mathscr{R}$, the product $\mathscr{R}\mathscr{R}(\vec{n},\alpha)\mathscr{R}^{-1}$ is a rotation through α about the axis with direction $\vec{n}' = \mathscr{R}\vec{n}$.

ii. Deduce from the result of **i.** the conjugacy classes of the group of three-dimensional rotations.

13. A quantum-mechanical problem with D_n symmetry

In the lecture, you saw that the energy-eigenfunctions of a quantum-mechanical problem in one spatial dimension with an even potential $V(x)$ are necessarily either even or odd. The present exercise relies on the same idea.

i. Consider the (x,y)-plane. Let $\mathscr{R}_n$ denote the two-dimensional rotation through an angle $2\pi/n$ (with $n \in \mathbb{N}^*$) around the origin and $\mathscr{S}_y$ denote the reflection across the x-axis.

Show that $\mathscr{R}_n$ and $\mathscr{S}_y$ with the usual composition of geometrical transformations generate a finite group D_n, which for $n \geq 2$ is the symmetry group of an n-sided regular polygon[1] centered on the origin and with one corner on the x-axis. (For $n = 1$, D_1 is the symmetry group of the figure consisting of two points at $x=x_0, y=y_0$ and $x=x_0, y=-y_0$ respectively, with $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^*$.) What is the order of D_n? Check that for every element $\mathscr{R} \in \mathrm{D}_n$, the identity $\mathscr{S}_y^{-1}\mathscr{R}\mathscr{S}_y = \mathscr{R}^{-1}$ holds.

ii. Consider the motion in two spatial dimensions — in the (x,y)-plane — of a particle in a potential $V(x,y)$ with D_n symmetry about the origin. If $|\psi\rangle$ denotes a state vector of the system and $\psi(x,y)$ the corresponding wave function in position representation, then $\hat{\mathscr{R}}|\psi\rangle$ denotes the state vector corresponding to the wave function ψ evaluated at the point $\mathscr{R}(x,y)$ with $\mathscr{R} \in \mathrm{D}_n$.

a) Let $|\psi_E\rangle$ denote an eigenstate of the Hamiltonian $\hat{H}$ of the system. Show that

$$|\psi_k\rangle = \sum_{p=1}^{n} \mathrm{e}^{2\mathrm{i}\pi kp/n}\big(\hat{\mathscr{R}}_n\big)^p|\psi_E\rangle \quad \text{for } k \in \{1,\dots,n\}$$

and

$$|\psi_\pm\rangle = \big(\hat{\mathbb{1}} \pm \hat{\mathscr{S}}_y\big)|\psi_E\rangle$$

are also eigenstates of $\hat{H}$, where $\hat{\mathbb{1}}$ denotes the identity operator on the Hilbert space of the system.

b) Show the orthogonality relations $\langle\psi_k|\psi_{k'}\rangle = 0$ for $k,k' \in \{1,\dots,n\}$ with $k \neq k'$ and $\langle\psi_+|\psi_-\rangle = 0$.

c) Show that $|\psi_k\rangle$ is eigenvector of $\hat{\mathscr{R}}_n$ and $|\psi_\pm\rangle$ eigenvector of $\hat{\mathscr{S}}_y$. What are the respective eigenvalues?

d) Why does a joint eigenbasis of $\hat{\mathscr{R}}_n$, $\hat{\mathscr{S}}_y$, and $\hat{H}$ exist if and only if $n \in \{1,2\}$? To answer this question calculate the action of $\hat{\mathscr{S}}_y$ on the state vector

$$|\psi_{k,\pm}\rangle = \sum_{p=1}^{n} \mathrm{e}^{2\mathrm{i}\pi kp/n}\big(\hat{\mathscr{R}}_n\big)^p\big(\hat{\mathbb{1}} \pm \hat{\mathscr{S}}_y\big)|\psi_E\rangle \quad \text{with } k \in \{1,\dots,n\}.$$

[1] OK, you may have difficulty picturing in your head the "digon" with $n = 2$. You may replace it with a non-square rectangle with its sides parallel to the coordinate axes, and let the length of the sides parallel to the y-axis go to 0.

Tutorial sheet 4

14. Center, centralizer and normalizer

Let $\mathcal{G}$ be a group. The *center* $Z(\mathcal{G})$ of $\mathcal{G}$ is the set of elements $g \in \mathcal{G}$ that commute with every element of $\mathcal{G}$:

$$Z(\mathcal{G}) = \{g \in \mathcal{G} \mid \forall g' \in \mathcal{G}, gg' = g'g\}.$$

If $\mathcal{S}$ denotes a subset of $\mathcal{G}$, the *centralizer* $C_{\mathcal{G}}(\mathcal{S})$ of $\mathcal{S}$ is the set of elements $g \in \mathcal{G}$ that commute with each element of $\mathcal{S}$:

$$C_{\mathcal{G}}(\mathcal{S}) = \{g \in \mathcal{G} \mid \forall g' \in \mathcal{S}, gg' = g'g\}.$$

Eventually, the *normalizer* $N_{\mathcal{G}}(\mathcal{S})$ of $\mathcal{S}$ is defined by

$$N_{\mathcal{G}}(\mathcal{S}) = \{g \in \mathcal{G} \mid g\mathcal{S} = \mathcal{S}g\}.$$

i. Show that $Z(\mathcal{G})$, $C_{\mathcal{G}}(\mathcal{S})$, and $N_{\mathcal{G}}(\mathcal{S})$ are subgroups of $\mathcal{G}$. How are $C_{\mathcal{G}}(\mathcal{S})$ and $N_{\mathcal{G}}(\mathcal{S})$ related when $\mathcal{S}$ consists of a single element?

ii. Show that $Z(\mathcal{G})$ is Abelian and normal in $\mathcal{G}$, and that $C_{\mathcal{G}}(\mathcal{S})$ is normal in $N_{\mathcal{G}}(\mathcal{S})$. Note that the centralizer is not necessarily normal in $\mathcal{G}$, nor Abelian.

iii. Show that if $\mathcal{H}$ is a subgroup of $\mathcal{G}$, then $\mathcal{H}$ is a normal subgroup of its normalizer $N_{\mathcal{G}}(\mathcal{H})$.

15. Pauli matrices

Consider the set consisting of the two-dimensional unit matrix and the Pauli matrices

$$\Sigma = \{\mathbb{1}_2, \sigma_1, \sigma_2, \sigma_3\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

i. Why is the set Σ not a group with respect to matrix multiplication? Extend Σ to a group $\mathcal{G}$ by closing the matrix multiplication.

ii. Calculate the center of $\mathcal{G}$ denoted by $Z(\mathcal{G})$ and identify it with a discrete group you already know.

iii. Determine the quotient group $\mathcal{G}/Z(\mathcal{G})$. Interpret and explain this result (which you may find confusing at first).

iv. Calculate all conjugacy classes of $\mathcal{G}$.

v. Calculate the centralizer of all elements of $\mathcal{G}$.

16. Group homomorphisms

Consider two groups $\mathcal{G}$ and $\mathcal{G}'$ with neutral elements e and e', respectively, and let $f : \mathcal{G} \to \mathcal{G}'$ be a homomorphism. Prove that the following statements hold:

i. $f(e) = e'$ and $f(g^{-1}) = \big(f(g)\big)^{-1}$;

ii. $\operatorname{im} f$ is a subgroup of $\mathcal{G}'$ and $\ker f$ is a normal subgroup of $\mathcal{G}$;

iii. more generally, f sends a subgroup $\mathcal{H}$ of $\mathcal{G}$ to a subgroup of $\mathcal{G}'$, and the "preimage" by f of a subgroup $\mathcal{H}'$ of $\mathcal{G}'$ is a subgroup of $\mathcal{G}$;

iv. f is injective if and only if $\ker f = \{e\}$.

17. Inner automorphisms

Let $\mathcal{G}$ be a group. For every $a \in \mathcal{G}$, one defines a mapping $\phi_a : \mathcal{G} \to \mathcal{G}$ by $g \mapsto \phi_a(g) = aga^{-1}$.

i. Check that ϕ_a is an isomorphism — it is called an *inner automorphism* of $\mathcal{G}$.

ii. Show that the set of inner automorphisms of $\mathcal{G}$ is a subgroup $\mathrm{Inn}(\mathcal{G})$ of the group of automorphisms of $\mathcal{G}$.

iii. Show that ϕ provides a group homomorphism from $\mathcal{G}$ onto $\mathrm{Inn}(\mathcal{G})$. What is the kernel of this homomorphism?

Tutorial sheet 5

18. Faithful representation

Let $\mathcal{G}$ be a group and $\mathscr{D}$ a representation of $\mathcal{G}$. Show that $\mathscr{D}$ is a faithful representation of the quotient group $\mathcal{G}/\ker\mathscr{D}$.

19. Representation of the Galilei group

In exercise **11.** you already encountered the Galilei group Gal(3).

i. Show that a 5-dimensional linear representation of Gal(3) consists of the matrices of the type

$$\begin{pmatrix} \mathscr{R} & \vec{v} & \vec{a} \\ \vec{0}^{\mathsf{T}} & 1 & \tau \\ \vec{0}^{\mathsf{T}} & 0 & 1 \end{pmatrix}, \tag{1}$$

where $\vec{0}^{\mathsf{T}}$ denotes a row vector with three zero entries, while $\vec{v}$ and $\vec{a}$ are two column vectors with three arbitrary real entries and $\mathscr{R}$ is a three-dimensional rotation matrix, $\mathscr{R} \in \mathrm{SO}(3)$.

ii. Is this representation faithful? Is it fully reducible? (As you will learn later, the — negative — answer reflects the fact that the Galilei group is not compact.)

20. A matrix representation of a well-known finite group

Consider the $n \times n$ matrix

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{2}$$

i.e. $T_{ij} = 1$ for all $j = i + 1$ and for $(i = m, j = 1)$, and otherwise $T_{ij} = 0$.

i. Construct a group $\tilde{\mathcal{G}}$ from T via matrix multiplication. The matrix group $\tilde{\mathcal{G}}$ is a representation of a finite group $\mathcal{G}$ which you already know; which one?

ii. With the help of Maschke's theorem we know that the representation $\tilde{\mathcal{G}}$ is completely reducible. What are all irreducible representations of $\tilde{\mathcal{G}}$? Calculate for this purpose the eigenspaces of T.

21. Symmetry in a classical many-body system

Consider a planar molecule, in the absence of external forces, with $N > 1$ interacting atoms at coordinates $\mathbf{x}_{i,0} + \mathbf{x}_i \equiv (x_{i,0} + x_i, y_{i,0} + y_i)$ for $i \in \{1, ..., N\}$, where $\mathbf{x}_{i,0} = (x_{i,0}, y_{i,0})$ is the equilibrium position of the i-th atom with mass m_i.

i. Explain why the part of the Hamilton function of the molecule relevant for the description of small oscillations of the atoms about their equilibrium positions may be written as

$$H = \frac{1}{2}\dot{\boldsymbol{X}}^{\mathsf{T}} M^2 \dot{\boldsymbol{X}} + \boldsymbol{X}^{\mathsf{T}} V \boldsymbol{X} \tag{3}$$

with $\boldsymbol{X} \equiv (x_1, y_1, x_2, y_2, ...)$. What is the significance of the matrices M and V?

ii. Show that if the molecule is invariant under the transformations $\boldsymbol{X} \to \boldsymbol{X'} = \mathscr{D}(g)^{-1}\boldsymbol{X}$, where $\mathscr{D}$ is a $2N$-dimensional representation of a group $\mathcal{G}$, then this implies the commutator $[V', \mathscr{D}'] = 0$ where $\mathscr{D}' = M\mathscr{D}M^{-1}$ and $V' = M^{-1}VM^{-1}$.

iii. A normal mode of the system is defined as a solution to the equations of motion in which all atoms oscillate with the same frequency. Show that for a normal mode, $M\boldsymbol{X}$ is an eigenvector of V' and that if $M\boldsymbol{X}$ is an eigenvector, then so is $\mathscr{D}'M\boldsymbol{X}$. Under which conditions is $\mathscr{D}'$ a reducible representation and if so, what do the invariant subspaces correspond to?

iv. Let us consider two examples of the system with identical particles ($m_i = m$ for all $i \in \{1, ..., N\}$), e.g. a benzene ring. The symmetry group $\mathcal{G}$ will be in either case the cyclic group C_N, only the matrix representation will be different. In the first case the generator of the representation, T_1, rotates each particle through the same angle, i.e.

$$\mathbf{x}_i \to \mathbf{x}_i' = \begin{pmatrix} \cos\frac{2\pi}{N} & \sin\frac{2\pi}{N} \\ -\sin\frac{2\pi}{N} & \cos\frac{2\pi}{N} \end{pmatrix} \mathbf{x}_i \tag{4}$$

In the second case, the generator of the representation, T_2, performs a cyclic permutation of the particles, $\mathbf{x}_i \to \mathbf{x}_i' = \mathbf{x}_{i+1}$ for $i \in \{1, ..., N-1\}$ and $\mathbf{x}_N \to \mathbf{x}_N' = \mathbf{x}_1$. What is the relation between these two representations? What does each of the two symmetries tell us about the matrix V? Into which irreducible representations do the representations generated by T_1 and T_2 split?

Tutorial sheet 6

22. Direct-product representation

Let $\mathscr{G}$ be a group and $\mathscr{D}$ an irreducible representation of $\mathscr{G}$. Show that $\mathscr{D} \otimes \mathscr{D}$ is an irreducible representation of the product group $\mathscr{G} \times \mathscr{G}$.

23. Möbius group

Consider the following nonlinear transformations in the complex plane:

$$z \to \frac{az + b}{cz + d}, \tag{1}$$

where a, b, c, d are complex numbers such that $ad \neq bc$.

i. Prove that the set of all such transformations forms a group (called the *Möbius group*) under composition. Why is the condition $ad \neq bc$ necessary?

ii. Explain why the following constraints define subgroups of the Möbius group:
a) $a = 1$, $b \in \mathbb{C}$, $c = 0$, $d = 1$; **b)** $|a| = 1$, $b = c = 0$, $d = 1$; **c)** $a \in \mathbb{R}^*$, $b = c = 0$, $d = 1$.
To which transformations in the complex plane do these subgroups correspond?

Remark: As opposed to these, the transformations with $c \neq 0$ do not preserve distance in the complex plane, yet they still preserve angles; such transformations are in general called conformal.

iii. Show that the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfy the same multiplication rule as the elements of the Möbius group. Do they provide a representation? Impose a suitable constraint on a, b, c, d such that resulting restricted set of matrices does provide a representation of the Möbius group.

Hint: The role of the constraint is to ensure that a given Möbius transformation maps to only one matrix.

24. Conjugate representation

Let $\mathscr{D}$ be a (matrix) representation of a group $\mathscr{G}$. For every $g \in \mathscr{G}$, $\mathscr{D}^*(g)$ denotes the complex conjugate of $\mathscr{D}(g)$.

i. Show that $\mathscr{D}^*$ is also a representation of $\mathscr{G}$.

ii. Assume that $\mathscr{D}$ and $\mathscr{D}^*$ are equivalent representations, i.e. there exists an invertible matrix C such that $\mathscr{D}^*(g) = C^{-1}\mathscr{D}(g)C$ for all $g \in \mathscr{G}$. Prove that if $\mathscr{D}$ is irreducible then there exists $\lambda \in \mathbb{C}$ such that $CC^* = \lambda\mathbb{1}$.

iii. Show that if $\mathscr{D}$ is unitary, then also $CC^\dagger = \mu\mathbb{1}$ with $\mu \in \mathbb{C}$. Moreover show that C may be chosen to be either symmetric or antisymmetric.

25. Symmetry group of a square

The symmetry group of a square is the dihedral group D_4. Determine its conjugacy classes and use them to find the number and dimensions of all irreducible representations of the group. Determine the character table of the group with the help of the orthogonality relations. Give the geometrical interpretation of all the irreducible representations.

Hint: The group D_4 has eight elements. You need to guess some of the representations in order to be able to use the orthogonality relations efficiently. The two one-dimensional and the two-dimensional representations are rather obvious.

Tutorial sheet 7

26. Regular representation

The *regular representation* of a finite group $\mathcal{G} = \{g_i\}$ is a $|\mathcal{G}|$-dimensional representation $\hat{\mathscr{D}}^{(\mathrm{r})}$ defined as follows. Given a basis $\{\mathrm{e}_i\}$ of the representation space $\mathscr{V}$, one first associates each group element g_i to the basis vector e_i for $i \in \{1, \ldots, |\mathcal{G}|\}$. For each $g \in \mathcal{G}$, the linear operator $\hat{\mathscr{D}}^{(\mathrm{r})}(g)$ on $\mathscr{V}$ maps the basis vector e_i to the basis vector e_j associated to the group element $g_j = gg_i$.[1]

i. Write down the matrices of the regular representations of the cyclic groups C_2, C_3 and C_4 and of the Klein group V_4.

ii. Determine the character $\chi^{(\mathrm{r})}$ of the regular representation.

Hint: Distinguish between the identity element e of $\mathcal{G}$ and the other elements of the group.

iii. Let $\{\hat{\mathscr{D}}^{(\alpha)}\}$ denote a set of inequivalent unitary irreducible representations of $\mathcal{G}$, with respective representation spaces $\{\mathscr{V}^{(\alpha)}\}$ and characters $\{\chi^{(\alpha)}\}$.

a) Show with the help of characters that the coefficient a_α in the decomposition $\hat{\mathscr{D}}^{(\mathrm{r})} = \bigoplus_\alpha a_\alpha \hat{\mathscr{D}}^{(\alpha)}$ of the regular representation equals the dimension of the representation space $\mathscr{V}^{(\alpha)}$.

b) Deduce from **a)** the equality $\sum_{\text{irreps. } \alpha} \left(\dim \mathscr{V}^{(\alpha)}\right)^2 = |\mathcal{G}|$.

27. Group representation from a representation of a quotient group

Let $\mathcal{N}$ be a normal subgroup of a group $\mathcal{G}$. A representation $\mathscr{D}_{\mathcal{G}/\mathcal{N}}$ of the quotient group $\mathcal{G}/\mathcal{N}$ can be *lifted* to a representation $\mathscr{D}_{\mathcal{G}}$ of the group $\mathcal{G}$ of the same dimension by defining

$$\mathscr{D}_{\mathcal{G}}(g) \equiv \mathscr{D}_{\mathcal{G}/\mathcal{N}}(g\mathcal{N}) \quad \forall g \in \mathcal{G}. \tag{1}$$

i. Check that Eq. (1) indeed defines a representation of $\mathcal{G}$. How are the characters of $\mathscr{D}_{\mathcal{G}/\mathcal{N}}$ and $\mathscr{D}_{\mathcal{G}}$ related?

ii. The symmetry group of a square D_4, which you already encountered in exercise **25.**, has a normal subgroup $\mathcal{N}$ consisting of the identity transformation and the rotation through 180° — this is actually the center of D_4.

a) Check that the quotient group $\mathrm{D}_4/\mathcal{N}$ is isomorphic to the Klein group V_4.

b) Write down the character table for V_4 and show that the characters of some of the irreducible representations of D_4 (exercise **25.**) can be lifted from those of the irreps. of V_4.

28. Symmetry group of a square (2)

The symmetry group of a square D_4 has a (normal) subgroup, consisting of the rotations that leave the square invariant, which is isomorphic to the group C_4.

i. What are the irreducible representations of this subgroup?

ii. In exercise **25.** you found that all one-dimensional (irreducible) representations of the group D_4 are real. This means that some of the irreps. of the normal subgroup C_4 are not irreps. of D_4. Explain why. What consequence can this have in a physical context?

[1] For the purist, the representation thus defined is the left regular representation of $\mathcal{G}$.

Tutorial sheet 8

29. Tensor product of irreducible representations

Prove the following results regarding the (irreducible) representations of a finite group $\mathcal{G}$. In **ii.**, $\hat{\mathscr{D}}^*$ denotes the conjugate representation of $\hat{\mathscr{D}}$, as introduced in exercise **24.**

i. The tensor product of an irreducible representation with a representation of dimension 1 is irreducible.

ii. Let $\hat{\mathscr{D}}_1$, $\hat{\mathscr{D}}_2$ and $\hat{\mathscr{D}}_3$ be three unitary irreducible representations. The number of times that $\hat{\mathscr{D}}_1^*$ is contained in (the Clebsch–Gordan series of) $\hat{\mathscr{D}}_2 \otimes \hat{\mathscr{D}}_3$ is equal to the number of times that $\hat{\mathscr{D}}_2^*$ is contained in $\hat{\mathscr{D}}_1 \otimes \hat{\mathscr{D}}_3$ and to the number of times that $\hat{\mathscr{D}}_3^*$ is contained in $\hat{\mathscr{D}}_1 \otimes \hat{\mathscr{D}}_2$.

iii. **a)** Let $\hat{\mathscr{D}}_1$, $\hat{\mathscr{D}}_2$ be two unitary irreducible representations with respective dimensions d_1, d_2 with $d_1 \geq d_2$. The tensor product $\hat{\mathscr{D}}_1 \otimes \hat{\mathscr{D}}_2$ contains no representation of dimension lower than d_1/d_2.

Hint: The result from **ii.** can be helpful.

b) As we shall see later in the lecture, a "spin j" (particle) corresponds to an irreducible representation (of the group of rotations) of dimension $2j+1$. Refresh your knowledge from your Quantum Mechanics lecture on the addition of two spins j_1 and j_2. Check that the lower bound on the total spin which you learned back then is compatible with the result from **iii.a)**.

30. Crystal-field splitting

In the lecture of November 28th, you heard of the octahedral group $\mathcal{O}$ with its five inequivalent irreducible representations $\hat{\mathscr{D}}^{(1)}$, $\hat{\mathscr{D}}^{(1')}$, $\hat{\mathscr{D}}^{(2)}$, $\hat{\mathscr{D}}^{(3)}$, $\hat{\mathscr{D}}^{(3')}$: these represent the rotations leaving invariant the electric field felt by an ion at a site surrounded by a regular octahedron of identical ions in an ideal crystal. Suppose now that the crystal is distorted, and more precisely that it is elongated along one of the (four) axes with threefold symmetry.

i. Convince yourself that the distortion reduces the rotational symmetry of the crystal to the dihedral group D_3. Which rotations of $\mathcal{O}$ are still present in D_3?

Hint: The three rotations through 180° that survive are in the conjugacy class in $\mathcal{O}$ with 6 elements.

ii. With respect to the three irreducible representations of D_3, which you already encountered several times, the irreps. of $\mathcal{O}$ may become reducible. Perform those reductions and discuss their meaning for the degeneracy of states with angular-momentum quantum number $\ell = 0, 1, 2, 3$.

31. Conjugate representation (2)

The conjugate representation $\mathscr{D}^*$ to a matrix representation $\mathscr{D}$ of a (finite) group $\mathcal{G}$ was introduced in exercise **24.**. In this exercise, we want to classify the possible relations between $\mathscr{D}^*$ and $\mathscr{D}$.

i. Show that if the character $\chi_{\mathscr{D}}(g)$ of $\mathscr{D}(g)$ is real-valued for all $g \in \mathcal{G}$, then $\mathscr{D}$ and $\mathscr{D}^*$ are equivalent — in which case one talks of a *self-conjugate representation*.

ii. In exercise **24.ii.**, you showed that if $\mathscr{D}$ is an irreducible self-conjugate representation, then there exists $\lambda \in \mathbb{C}$ such that the similarity matrix C between every $\mathscr{D}(g)$ and $\mathscr{D}^*(g)$ obeys $CC^* = \lambda \mathbb{1}$. Show that you can choose $\lambda = \pm 1$.

iii. Assume further that $\mathscr{D}$ is unitary, so that you know from exercise **24.iii.** that the similarity matrix C can be taken to be either symmetric or antisymmetric: $C^{\mathrm{T}} = \pm C$.

a) Prove that if the character $\chi_{\mathscr{D}}$ is real-valued, then

$$\frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}}\chi_{\mathscr{D}}(g^2) = \pm 1, \tag{1}$$

the sign on the right hand side being the same as in the relation between C^{T} and C.

b) Show that if $\mathscr{D}$ is real, i.e. $\mathscr{D}^*(g) = \mathscr{D}(g)$ for all $g \in \mathcal{G}$, then Eq. (1) holds with $+1$ on the right hand side.

The converse is true: if the quantity on the left hand side of Eq. (1) equals $+1$, then $\mathscr{D}$ is real.[1]

iv. One can further prove that if $\mathscr{D}$ and $\mathscr{D}^*$ are not equivalent, then the character $\chi_{\mathscr{D}}$ is complex-valued and such that

$$\frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}}\chi_{\mathscr{D}}(g^2) = 0. \tag{2}$$

In exercise **25.** and **28.** you found the characters of the irreducible representations of the dihedral group D_4 and the cyclic group C_4. Compute the quantity on the left hand side of Eq. (1) / (2) for each irrep. of D_4 or C_4 and tell which ones are real, self-conjugate but not real ("pseudo-real"), or not self-conjugate ("complex").

[1] If you wish to prove it, show that there exists a unitary and symmetric matrix A such that $A^2 = C$ and consider the representation $\mathscr{D}' = A\mathscr{D}A^{-1}$.

Tutorial sheet 9

32. Symmetrizer and antisymmetrizer

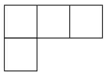
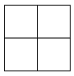
In the lecture on December 1st, the symmetrizer and antisymmetrizer

$$\mathscr{S} \equiv \frac{1}{n!}\sum_{\sigma\in\mathrm{S}_n}\sigma \quad\text{and}\quad \mathscr{A} \equiv \frac{1}{n!}\sum_{\sigma\in\mathrm{S}_n}\varepsilon(\sigma)\sigma\,, \tag{1}$$

elements of the group algebra $\mathbb{C}\mathrm{S}_n$ of the symmetric group S_n, were introduced.
Let $\tau \in \mathrm{S}_n$ be a permutation. Compute first $\tau\mathscr{S}$ and $\tau\mathscr{A}$, and then $\mathscr{S}^2$, $\mathscr{A}^2$, $\mathscr{S}\mathscr{A}$ and $\mathscr{A}\mathscr{S}$.

33. Representations of S_n

The following Young diagrams are associated to two irreps. of the symmetric group S_4:

 ,  . (2)

Compute their outer product and give the dimensions of all irreducible representations involved.

34. Symplectic group

Let $\mathbb{1}_n$ denote the $n\times n$ identity matrix, where $n \in \mathbb{N}^*$.

i. Show that the set of $2n\times 2n$ matrices M with real entries satisfying

$$M^\mathsf{T}\begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0\end{pmatrix}M = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0\end{pmatrix} \tag{3}$$

is a group — called *symplectic group* and denoted[1] $\mathrm{Sp}(2n,\mathbb{R})$ — for the usual matrix product. How many real parameters are needed to characterize an element of this group?
Hint: Find a relation between the inverse matrix M^{-1} and the transposed matrix M^T.

ii. Check that the matrices of $\mathrm{Sp}(2n,\mathbb{R})$ preserve the scalar product

$$\boldsymbol{x}\cdot\boldsymbol{y} \equiv \sum_{a=1}^{n}(x_a y_{a+n} - x_{a+n}y_a) \tag{4}$$

where $\boldsymbol{x},\boldsymbol{y}$ are $2n$-dimensional real vectors with coordinates x_a, y_a. Do you see where in physics the coordinate transformations realized by the symplectic matrices play a role? (*hint*: classical mechanics)

35. Exponential of a matrix

Let A and B be two complex $n\times n$ matrices. Show the following results:

i. If A and B commute, then $\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{A+B}$.

ii. $\left(\mathrm{e}^A\right)^{-1} = \mathrm{e}^{-A}$.

iii. If B is regular, then $B\,\mathrm{e}^A B^{-1} = \mathrm{e}^{BAB^{-1}}$.

iv. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A, then the eigenvalues of e^A are $\mathrm{e}^{\lambda_1}, \dots, \mathrm{e}^{\lambda_n}$, so that $\det\mathrm{e}^A = \mathrm{e}^{\mathrm{Tr}\,A}$.

[1] The group is unfortunately sometimes denoted $\mathrm{Sp}(n,\mathbb{R})$...

Tutorial sheet 10

36. Lie algebras of classical groups

i. Let $\mathcal{G}$ be a matrix Lie group and $\mathfrak{g}$ the space of matrices such that $M \in \mathfrak{g}$ implies $\mathrm{e}^M \in \mathcal{G}$.

Show that **a)** $\mathfrak{gl}(n)$, **b)** $\mathfrak{sl}(n)$, **c)** $\mathfrak{o}(n) = \mathfrak{so}(n)$, **d)** $\mathfrak{u}(n)$, **e)** $\mathfrak{su}(n)$, respectively associated to the groups GL(n), SL(n), O(n) or SO(n), U(n), SU(n), consist of **a)** arbitrary $n \times n$ matrices, **b)** traceless matrices, **c)** antisymmetric matrices, **d)** antihermitian matrices, **e)** traceless antihermitian matrices.

ii. For each of the above cases, check that the characteristic property (tracelessness, antisymmetry…) of the matrices of $\mathfrak{g}$ is preserved by the commutator, thereby ensuring that $\mathfrak{g}$ is a Lie algebra.

37. An example of the non-surjectivity of the exponential map

Consider the non-compact group $\mathrm{SL}(2,\mathbb{R})$: its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ consists of traceless 2×2-matrices with real entries.

i. For any $M \in \mathfrak{sl}(2,\mathbb{R})$, show that $\mathrm{Tr}\, M^{2n} = 2(-\det M)^n$ and $\mathrm{Tr}\, M^{2n+1} = 0$ for all $n \in \mathbb{N}$.

Hint: Consider the characteristic polynomial of M.

ii. Deduce from **i.** that $\mathrm{Tr}\, \mathrm{e}^M \geq -2$ for all $M \in \mathfrak{sl}(2,\mathbb{R})$. Give an example of element of $\mathrm{SL}(2,\mathbb{R})$ with a trace strictly smaller than -2 and conclude.

38. Relation between SO(3) and SU(2)

i. The group SO(3) consists of the real 3×3-matrices with determinant 1 that preserve the usual (Euclidean) scalar product, i.e. it describes rotations in $\mathbb{R}^3$. Let $\mathscr{R}_{\vec{n}}(\alpha)$ denote the rotation through the angle α about the axis with unit vector $\vec{n}$.

a) As a warm up, convince yourself that the rotations $\mathscr{R}_{\vec{n}}(\alpha)$ and $\mathscr{R}_{-\vec{n}}(-\alpha)$ are identical and derive the Rodrigues formula

$$\mathscr{R}_{\vec{n}}(\alpha)\vec{x} = (\cos\alpha)\vec{x} + (1-\cos\alpha)(\vec{n}\cdot\vec{x})\vec{n} + (\sin\alpha)\vec{n}\times\vec{x} \qquad \forall \vec{x} \in \mathbb{R}^3. \tag{1}$$

Hint: Decompose $\vec{x}$ in two components parallel and perpendicular to $\vec{n}$.

b) Associate to the rotation $\mathscr{R}_{\vec{n}}(\alpha)$ the vector $\boldsymbol{u}$ with 4 components $\boldsymbol{u} \equiv (\vec{u} \equiv \vec{n}\cos\frac{\alpha}{2}, u_4 \equiv \sin\frac{\alpha}{2})$. Show that $\boldsymbol{u}$ belongs to the unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4$, i.e. that the Euclidean norm $[\vec{u}^2 + (u_4)^2]^{1/2}$ of $\boldsymbol{u}$ equals 1. What happens to $\mathscr{R}_{\vec{n}}(\alpha)$ and to $\boldsymbol{u}$ when you add to α an odd multiple of 2π?

ii. The group SU(2) consists of the unitary 2×2-matrices with determinant 1.

a) (Reminder of your Quantum Mechanics lecture) Check that for every point $\boldsymbol{u} \in \mathbb{S}^3$, the matrix

$$U = u_4\,\mathbb{1}_2 - \mathrm{i}\,\vec{u}\cdot\vec{\sigma} \tag{2}$$

is in SU(2), where $\mathbb{1}_2$ is the unit 2×2-matrix while $\vec{\sigma}$ is a "vector" whose entries are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Conversely, show that every matrix of SU(2) can be written in the form (2). Deduce therefrom that SU(2) and $\mathbb{S}^3$ are in bijection to each other. By "exporting" the group law of SU(2) to $\mathbb{S}^3$, one defines a group structure on the latter, which is then isomorphic to SU(2).

b) Let $\vec{n}$ be a unit vector of $\mathbb{R}^3$ (one may write that $\vec{n}$ belongs to the sphere $\mathbb{S}^2$) and $\alpha \in [0, 2\pi]$. Show the identity

$$U_{\vec{n}}(\alpha) \equiv \mathrm{e}^{-\mathrm{i}\alpha\vec{n}\cdot\vec{\sigma}/2} = \left(\cos\frac{\alpha}{2}\right)\mathbb{1}_2 - \mathrm{i}\left(\sin\frac{\alpha}{2}\right)\vec{n}\cdot\vec{\sigma}, \tag{4}$$

which defines the matrix $U_{\vec{n}}(\alpha)$.

iii. Homomorphism between SU(2) and SO(3)

a) To each vector $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ one associates a traceless Hermitian matrix X according to

$$X \equiv \vec{x}\cdot\vec{\sigma} = \begin{pmatrix} x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & -x_3 \end{pmatrix}. \tag{5}$$

Show that the correspondence is bijective, i.e. that to any traceless Hermitian matrix X one can associate a unique vector $\vec{x} \in \mathbb{R}^3$. (*hint*: trace!)

b) Given a unitary matrix $U \in \mathrm{SU}(2)$, one maps the matrix X of Eq. (5) to the matrix $X' = UXU^\dagger$. This defines a linear transformation $\vec{x} \mapsto \vec{x}' = \mathcal{T}(U)\vec{x}$ such that $X' = \vec{\sigma}\cdot\vec{x}'$. Show that $\mathcal{T}(U)$ is a rotation.

Hint: You may either show that $\mathcal{T}(U)$ is an isometry — preserving $|\vec{x}|$ — with determinant 1, or compute X' by introducing an explicit form for U.

c) One can even show — if you did not use that result in **b)** — that $\mathcal{T}(U_{\vec{n}}(\alpha))$ is the rotation $\mathscr{R}_{\vec{n}}(\alpha)$. Show that $\mathcal{T}$ is a group homomorphism from SU(2) onto SO(3). Is it an isomorphism? If no, what is its kernel?

d) Summarize for yourself what you have learned of the relationships between SU(2), $\mathbb{S}^3$ and SO(3).

Tutorial sheet 11

39. Lie bracket of generators of $\mathfrak{so}(n)$

The Lie algebra $\mathfrak{so}(n)$ of the special orthogonal group $\mathrm{SO}(n)$ is the space of traceless antisymmetric $n \times n$ matrices, where $n \geq 2$. (Troughout this exercise, the position of the indices is irrelevant.)

i. Check that a basis of generators (in the physicists' convention) consists of matrices T^{ab}, with $1 \leq a < b \leq n$, whose ij-entry is $(T^{ab})_{ij} = -\mathrm{i}(\delta^a_i \delta^b_j - \delta^a_j \delta^b_i)$, where δ^k_l is the usual Kronecker symbol. Show that the Lie bracket of two such generators is

$$[T^{ab}, T^{cd}] = -\mathrm{i}\big(\delta^{bc}T^{ad} + \delta^{ad}T^{bc} - \delta^{bd}T^{ac} - \delta^{ac}T^{bd}\big). \tag{1}$$

ii. Show that in the case $n = 3$ you recover results from the lecture.

iii. Denote X a vector with n real components x^j. Given a matrix $O \in \mathrm{SO}(n)$ close to the identity matrix $\mathbb{1}_n$, the transformation $X \to OX \equiv X'$ yields $X' = X + \delta X$ where the components δx^j of δX are "small". Show that one may write

$$\delta x^j = -\frac{\mathrm{i}}{2}\omega_{ab}T^{ab}x^j \quad \text{with} \quad T^{ab} \equiv -\mathrm{i}\left(x^a \frac{\partial}{\partial x^b} - x^b \frac{\partial}{\partial x^a}\right) \tag{2}$$

and with ω_{ab} antisymmetric (and traceless). Convince yourself that the differential operators T^{ab} defined in Eq. (2) are in one-to-one correspondence with the matrices T^{ab} of question **i.**

40. SU(1,1) and SL$(2, \mathbb{R})$

i. The group SU(1,1) consists of the complex 2×2 matrices U with determinant 1 such that $U^\dagger \eta U = \eta$, where $\eta \equiv \mathrm{diag}(1, -1)$.

a) What is the dimension of SU(1,1)?

b) Which equation does an element X of the Lie algebra $\mathfrak{su}(1,1)$ obey? What does that equation imply for the matrix elements of X? Prove that one may write a basis of $\mathfrak{su}(1,1)$ in terms of the Pauli matrices and compute their commutation relations.

c) Is $\mathfrak{su}(1,1)$ isomorphic[1] to the algebra $\mathfrak{so}(3)$?

ii. Consider now the special linear group $\mathrm{SL}(2, \mathbb{R})$.

a) Recall its definition and give its dimension. How is its Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ defined? Give a basis in terms of Pauli matrices.

b) Show that the two Lie algebras $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2, \mathbb{R})$ are isomorphic.

41. Two-dimensional representation of U(1)

Consider the map $\mathscr{D}^{(m)}$ from U(1) into SO(2) defined by

$$\mathscr{D}^{(m)}(\mathrm{e}^{\mathrm{i}\alpha}) = \begin{pmatrix} \cos(m\alpha) & \sin(m\alpha) \\ -\sin(m\alpha) & \cos(m\alpha) \end{pmatrix}. \tag{3}$$

For which values of m is $\mathscr{D}^{(m)}$ a representation of the group U(1)? With which physical quantity would you associate the number m?

[1] A homomorphism of Lie algebras is a bijective linear application between the underlying vector spaces that preserves the Lie brackets.

42. A property of $\mathfrak{so}(4)$

In exercise **39.** you saw a family of generators of $\mathfrak{so}(n)$ and their Lie brackets. Take now $n = 4$ and define (beware the signs!)

$$A_1 \equiv \frac{1}{2}(T^{12} - T^{34}) \quad , \quad A_2 \equiv \frac{1}{2}(T^{13} + T^{24}) \quad , \quad A_3 \equiv \frac{1}{2}(T^{14} - T^{23}) \tag{4}$$

and

$$B_1 \equiv \frac{1}{2}(T^{12} + T^{34}) \quad , \quad B_2 \equiv \frac{1}{2}(-T^{13} + T^{24}) \quad , \quad B_3 \equiv \frac{1}{2}(T^{14} + T^{23}). \tag{5}$$

Compute the commutators $[A_i, A_j]$, $[B_i, B_j]$, and $[A_i, B_j]$. How would you be tempted to interpret your findings? (You actually know a physics problem with a "hidden" SO(4) symmetry, which will be the topic of a later exercise.)

We wish you a merry Christmas and a happy new year!

Tutorial sheet 12

43. SO(3) matrices in the spherical basis

Consider the spin-1 irreducible representation of SU(2).

i. Write down the generators $\mathscr{J}_1^{(3)}, \mathscr{J}_2^{(3)}, \mathscr{J}_3^{(3)}$ in the "spherical" basis $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$ consisting of the eigenvectors of $\mathscr{J}_3^{(3)}$ (*hint*: write down first the matrices $\mathscr{J}_+^{(3)}, \mathscr{J}_-^{(3)}$).

ii. Show that Wigner's small d-matrix reads

$$d^1(\psi) \equiv \mathrm{e}^{-\mathrm{i}\psi\mathscr{J}_2^{(3)}} = \begin{pmatrix} \frac{1+\cos\psi}{2} & -\frac{\sin\psi}{\sqrt{2}} & \frac{1-\cos\psi}{2} \\ \frac{\sin\psi}{\sqrt{2}} & \cos\psi & -\frac{\sin\psi}{\sqrt{2}} \\ \frac{1-\cos\psi}{2} & \frac{\sin\psi}{\sqrt{2}} & \frac{1+\cos\psi}{2} \end{pmatrix}. \tag{1}$$

44. Standard components of a vector operator

The so-called *standard components* of a vector operator $\hat{\vec{V}}$ are defined as

$$\hat{V}_1^{(1)} \equiv -\frac{1}{\sqrt{2}}(\hat{V}_x + \mathrm{i}\hat{V}_y) \quad , \quad \hat{V}_0^{(1)} \equiv \hat{V}_z \quad , \quad \hat{V}_{-1}^{(1)} \equiv \frac{1}{\sqrt{2}}(\hat{V}_x - \mathrm{i}\hat{V}_y), \tag{2}$$

where $\hat{V}_x, \hat{V}_y, \hat{V}_z$ are the Cartesian components of $\hat{\vec{V}}$.

Starting from the standard components $\hat{V}_m^{(1)}$, $\hat{W}_{m'}^{(1)}$ of two vector operators $\hat{\vec{V}}$, $\hat{\vec{W}}$, one defines operators

$$\big(\hat{\vec{V}} \otimes \hat{\vec{W}}\big)_M^{(J)} = \sum_{m,m'} C_{1,1;m,m'}^{J,M} V_m^{(1)} W_{m'}^{(1)} \tag{3}$$

where the numbers $C_{1,1;m,m'}^{J,M} \equiv \langle 1,1;m,m' | J,M\rangle$ are the Clebsch–Gordan coefficients relevant to the addition of two spins 1.[1]

i. Show that $\big(\hat{\vec{V}} \otimes \hat{\vec{W}}\big)_0^{(0)}$ is proportional to the scalar product $\hat{\vec{V}} \cdot \hat{\vec{W}}$ of the vector operators.

ii. Show that the three operators $\big(\hat{\vec{V}} \otimes \hat{\vec{W}}\big)_M^{(1)}$ are proportional to the three standard components of the vector operator $\hat{\vec{V}} \times \hat{\vec{W}}$.

iii. Express the five operators $\big(\hat{\vec{V}} \otimes \hat{\vec{W}}\big)_M^{(2)}$ in terms of $\hat{V}_z$, $\hat{V}_\pm \equiv \hat{V}_x \pm \mathrm{i}\hat{V}_y$, $\hat{W}_z$, and $\hat{W}_\pm \equiv \hat{W}_x \pm \mathrm{i}\hat{W}_y$.

45. Isospin and reaction rates

"Isospin" is an approximate SU(2) symmetry of the strong interaction: the proton and the neutron form a doublet (**2**-irrep., isospin $\frac{1}{2}$) where the proton (p) is the state $|j,m\rangle = |\frac{1}{2},\frac{1}{2}\rangle$ and the neutron (n) the state $|\frac{1}{2},-\frac{1}{2}\rangle$. When they bind together, a proton and a neutron form a deuteron (d), with isospin 0 (an "iso-scalar").

In proton-deuteron scattering, one observes (among others) the following two channels:

$$p + d \to \pi^0 + {}^3\mathrm{He} \quad , \quad p + d \to \pi^+ + {}^3\mathrm{H}. \tag{4}$$

Both pions in these reactions are elements of an "iso-triplet" (**3**-irrep., isospin 1) with respectively $\pi^+ = |1,1\rangle$ and $\pi^0 = |1,0\rangle$.[2] In turn, the ^{3}He and ^{3}H nuclei form an iso-doublet: ${}^3\mathrm{He} = |\frac{1}{2},\frac{1}{2}\rangle$, ${}^3\mathrm{H} = |\frac{1}{2},-\frac{1}{2}\rangle$.

[1] See e.g. `http://www-pdg.lbl.gov/2017/reviews/rpp2017-rev-clebsch-gordan-coefs.pdf` for a table with these coefficients.

[2] The $|1,-1\rangle$-state is the negatively charged π^-, which plays no role in this exercise

i. Using the fact that the relevant Hamilton operator is an iso-scalar — which means that isospin is a symmetry of the system —, express the transition amplitudes from the initial to the final states of reactions (4) as the product of a process-dependent coefficient controlled by the symmetry group and a matrix element which you cannot explicitly compute.

ii. Compute the ratio of the transition probabilities $\sigma(p + d \to \pi^0 + {}^3\text{He})$ and $\sigma(p + d \to \pi^+ + {}^3\text{H})$.

Tutorial sheet 13

46. Theories for a classical vector field

i. Consider a (4-)vector field with components $A^\mu(\mathsf{x})$, described by the Lagrange density

$$\mathscr{L}_0 = -\frac{1}{2}(\partial^\mu A^\nu)(\partial_\mu A_\nu - \partial_\nu A_\mu), \tag{1}$$

where the 4 components A^ν are to be considered as independent fields.

a) Write down the corresponding Euler–Lagrange equations. For the sake of brevity, you may introduce the notation $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$.

b) The Lagrange density (1) is invariant under the transformations $A^\mu \to A'^\mu = A^\mu + \varepsilon^\mu$ where the ε^μ are the components of an arbitrary, time- and space-independent 4-vector. Derive the Noether current J^μ associated with this symmetry. What is the conservation equation $\partial_\mu J^\mu = 0$ equivalent to?

ii. Consider now a vector field with the Lagrange density

$$\mathscr{L}_{\text{st.}} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{2}$$

and write down the corresponding Euler–Lagrange equations. What do you notice?

iii. How do the Lagrange densities (1) and (2) behave under the transformation $A^\mu \to A'^\mu = A^\mu + \partial^\mu\chi$, with χ an arbitrary (differentiable) function of the space-time coordinates? Is the group of transformations of that type a symmetry of the corresponding physical systems?

47. Isospin symmetry

i. Consider a theory for two Dirac spinor-fields ψ_p, ψ_n (standing respectively for a proton and a neutron) with the Lagrange density

$$\mathscr{L}_N = \bar\psi_p\big(\mathrm{i}\gamma^\mu\partial_\mu - m_p\big)\psi_p + \bar\psi_n\big(\mathrm{i}\gamma^\mu\partial_\mu - m_n\big)\psi_n, \tag{3}$$

where the γ^μ, $\mu = 0,1,2,3$ are Dirac matrices, while the adjoint spinors $\bar\psi_p$, $\bar\psi_n$ are to be treated as fields independent of ψ_p, ψ_n.

a) For the sake of completeness, derive the equations of motions obeyed by ψ_p and ψ_n. Are they known to you?

b) If $m_p \neq m_n$, the Lagrange density (3) is invariant under a "global" U(1) × U(1) symmetry whose transformations are of the form

$$\begin{cases} \psi_p \to \psi'_p = \mathrm{e}^{-\mathrm{i}\Lambda_p}\,\psi_p \quad , \quad \bar\psi_p \to \bar\psi'_p = \mathrm{e}^{\mathrm{i}\Lambda_p}\,\bar\psi_p \\ \psi_n \to \psi'_n = \mathrm{e}^{-\mathrm{i}\Lambda_n}\,\psi_n \quad , \quad \bar\psi_n \to \bar\psi'_n = \mathrm{e}^{\mathrm{i}\Lambda_n}\,\bar\psi_n \end{cases} \tag{4}$$

where Λ_p and Λ_n are time- and space-independent real constants. Derive the corresponding Noether currents. Can you guess what the associated conserved charges are?

ii. Assume now that $m_p = m_n$.

a) Introducing the "two-dimensional vector" with Dirac-spinor entries

$$\Psi \equiv \begin{pmatrix}\psi_p \\ \psi_n\end{pmatrix}$$

and its adjoint $\bar\Psi \equiv \big(\bar\psi_p\ \ \bar\psi_n\big)$, show that one can recast the Lagrange density (3) in the form $\mathscr{L}_N = \bar\Psi D\Psi$

where D is a "2×2" matrix proportional to $\mathbb{1}_2$, and that it is now invariant under transformations $\Psi \to \Psi' = \mathcal{U}\Psi$, $\bar{\Psi} \to \bar{\Psi}' = \bar{\Psi}\mathcal{U}^\dagger$ with a time- and space-independent $\mathcal{U} \in \mathrm{U}(2)$.

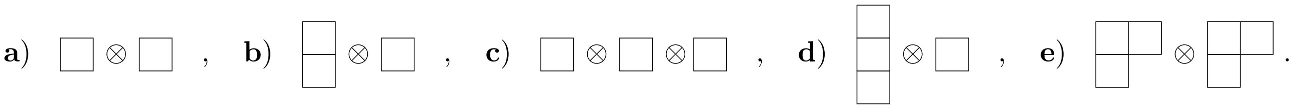
b) Writing $\mathcal{U}$ as $\mathrm{e}^{\mathrm{i}\theta}U$ with $\mathrm{e}^{\mathrm{i}\theta} \in \mathrm{U}(1)$ and $U \in \mathrm{SU}(2)$, write down infinitesimal transformations of the fields Ψ and $\bar{\Psi}$ (*hint*: write the SU(2) matrix in exponential form) and derive the Noether currents associated with the U(1) and SU(2) symmetries. SU(2) is the so-called "isospin symmetry" of the system (cf. exercise **45**).

If you have some notions of particle physics, can you guess what the conserved charge associated with the U(1) part corresponds to?

This exercise will be continued in exercise **49.**.

48. Young diagrams and representations of SU(N)

i. Compute the following products of Young diagrams:



ii. Viewing □ as the defining ($\boldsymbol{N}$) representation of SU(N), compute the dimensions of all irreducible representations involved in question **i.** for $N = 2, 3, 4$ and write down explicitly the corresponding Clebsch–Gordan series in the form $\boldsymbol{N} \otimes \boldsymbol{N} = \cdots$, and so on.
(For $N = 2$, you should be able to double-check the results with your knowledge from the "SU(2)-SO(3)" lecture; for $N = 3$, those with some knowledge of the quark model may recognize part of the results.)

49. Isospin symmetry (2)

i. Consider now a theory with 3 real scalar fields π_1, π_2, π_3, collectively denoted by a 3-vector $\vec{\pi}$, described by the Lagrange density

$$\mathscr{L}_\pi = -\frac{1}{2}\big(\partial_\mu\vec{\pi}\big)\cdot\big(\partial^\mu\vec{\pi}\big) - \frac{1}{2}m_\pi^2\,\vec{\pi}^2. \tag{5}$$

Check that this Lagrange density is invariant under transformations $\vec{\pi} \to \vec{\pi}' = O\vec{\pi}$ with a time- and space-independent $O \in \mathrm{SO}(3)$ and derive the corresponding Noether currents. Why is the present invariance possibly the "same" isospin symmetry as in exercise **47.ii.**?

ii. The fields Ψ introduced in **47.ii.a)** and $\vec{\pi}$ are coupled together by a term

$$\mathscr{L}_{N-\pi} = \mathrm{i}g\big(\bar{\Psi}\gamma^5\vec{\sigma}\Psi\big)\cdot\vec{\pi} \tag{6}$$

where $\vec{\sigma}$ denotes as usual a 3-vector whose entries are the Pauli matrices, while γ^5 is a matrix that acts on the spinorial degrees of freedom of the fields ψ_p, ψ_n and plays no role in the following.

a) Compute the behavior of the term $\bar{\Psi}\gamma^5\vec{\sigma}\Psi$ under infinitesimal isospin transformations (*hint*: you may need the identity $[\sigma_i, \alpha_j\sigma_j] = 2\mathrm{i}\alpha_j\sum_k \epsilon_{ijk}\sigma_k$) and deduce therefrom the invariance of the interaction Lagrange density (6) under isospin.

b) Derive the Noether current(s) associated with isospin symmetry for the theory with Lagrange density

$$\mathscr{L}_{N+\pi} = \mathscr{L}_N + \mathscr{L}_\pi + \mathscr{L}_{N-\pi}. \tag{7}$$

c) The particle-physics fans may rewrite the interaction term (6) in terms of the proton and neutron fields ψ_p, ψ_n and of the physical pions (π^+, π^-, π^0) given by

$$\pi_1 = \frac{1}{\sqrt{2}}(\pi^+ + \pi^-) \quad , \quad \pi_2 = -\frac{\mathrm{i}}{\sqrt{2}}(\pi^+ - \pi^-) \quad , \quad \pi_3 = \pi^0.$$

Tutorial sheet 14 ("Mock exam")

50. Dihedral group D_4

Let D_4 denote the dihedral group of order 8, i.e. the symmetry group of a square.

i. You might be tempted to consider that D_4 can be generated by a rotation r_4 through 90° and a reflexion s. Instead of that, explain how the 4 elements $\{s, sr_4, sr_4^2, sr_4^3\}$ can be viewed as rotations through 180°.

ii. The group D_4 consists of 5 conjugacy classes. Can you deduce from this information the number of irreducible representations of D_4 and their respective dimensions? Explain your result.

iii. Assume that the square defines the (x, y)-plane and consider a system of coordinates (x, y, z) such that the square vertices sit at $(\pm 1, \pm 1, 0)$. Construct a three-dimensional representation of D_4 and compute the corresponding characters. Is this representation reducible?

51. Coulomb potential

Consider the motion of a particle in a Coulomb potential, described by the Lagrange function

$$\mathcal{L} = \frac{m}{2}\dot{\vec{r}}^2 + \frac{k}{|\vec{r}|}, \tag{1}$$

where m is the mass of the particle and $k > 0$ some coupling constant.

i. What are the corresponding equations of motion?

ii. Consider the two kinds of infinitesimal transformations

$$\vec{r} \to \mathcal{T}_1(\vec{r}) = \vec{r} + \delta O\,\vec{r}, \tag{2}$$

where δO is an infinitesimal anti-symmetric 3×3 matrix, $\delta O^{\mathsf{T}} = -\delta O$, and

$$\vec{r} \to \mathcal{T}_2(\vec{r}) = \vec{r} + 2(\delta\vec{a}\cdot\vec{r})\vec{p} - (\delta\vec{a}\cdot\vec{p})\vec{r} - (\vec{r}\cdot\vec{p})\delta\vec{a}, \tag{3}$$

where $\delta\vec{a}$ is an infinitesimal vector of $\mathbb{R}^3$ and $\vec{p}$ the (linear) momentum of the particle.

a) Show that both infinitesimal transformations correspond to symmetries of the Lagrange function (1).
Hint: Expand $\mathcal{L}$ to first order in δO and $\delta\vec{a}$.

b) To which symmetry group does transformation (2) correspond?

iii) According to Noether's theorem, one can associate conserved quantities to both symmetries. Show that these quantities are

$$\vec{L} = \vec{r} \times \vec{p} \quad , \quad \vec{A} = \vec{p} \times (\vec{r} \times \vec{p}) - \frac{mkr}{|\vec{r}|}, \tag{4}$$

respectively. Explain their physical significance.

52. Heisenberg group

Let $\hat{x}$, $\hat{p}$ denote the position and momentum operators in the x-direction and $\hat{\mathbb{1}}$ be the identity operator.

i. Show that the three operators $\{\hat{x}, \hat{p}, \hat{\mathbb{1}}\}$ form a Lie algebra (the Heisenberg algebra), which is closed under taking commutators. Give its commutation relations and the structure constants.

ii. Consider the three-dimensional upper triangular matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with } a, b, c \in \mathbb{R}. \tag{5}$$

Show that the matrices of this type constitute a Lie group, often referred to as the *Heisenberg group*. Of which form are the matrices of the corresponding Lie algebra? Show that the latter is isomorphic to that considered in **i.**.

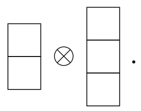
iii. Is the Heisenberg group compact?

53. Irreducible representations of SU(5)

i. Considering as usual that $\square$ stands for the defining (**5**) representation of SU(5), which Young diagrams are associated to the conjugate representation $\bar{\mathbf{5}}$ and to the adjoint representation? What is the dimension of the latter?

ii. Which (simple) Young diagram corresponds to the **10** representation? Inspiring yourself from the "correspondence" between the **5** and the $\bar{\mathbf{5}}$, can you then guess which Young diagram stands for the $\bar{\mathbf{10}}$ representation?

iii. Compute the outer product of the Young diagrams consisting of single columns with two resp. three boxes:



Viewing these diagrams as irreducible representations of SU(5), compute the dimensions of all irreducible representations involved and write down explicitly the corresponding Clebsch–Gordan series in the form $\mathbf{?} \otimes \mathbf{??} = \cdots$.

The group SU(5) was proposed as gauge group of a "grand unified theory" (GUT) encompassing the strong and electroweak interactions of particle physics (cf. Georgi–Glashow model).