Tutorial sheets 8 & 9

8. Two-level system coupled to a bath of harmonic oscillators

Consider a two-level system $\mathcal{S}$ with the Hamiltonian

$$\hat{\mathsf{H}}_{\mathcal{S}} = -\frac{\hbar\Omega}{2}\hat{\sigma}_z \tag{1}$$

with σ_z the usual diagonal Pauli matrix. The eigenstates of $H_{\mathcal{S}}$ will be denoted by $|0\rangle$ and $|1\rangle$. $\mathcal{S}$ couples to an environment $\mathcal{R}$ consisting (cf. lecture of June 9) of harmonic oscillators with free Hamiltonian

$$\hat{\mathsf{H}}_{\mathcal{R}} = \sum_j \hbar\omega_j \hat{\mathsf{b}}_j^\dagger \hat{\mathsf{b}}_j, \tag{2}$$

where for simplicity the zero-point energy of every single oscillator has been discarded. $\mathcal{R}$ is assumed to remain all along in a stationary state described by a statistical operator $\hat{\rho}^0_{\mathcal{R}}$ that is diagonal in the basis of eigenstates of $\hat{\mathsf{H}}_{\mathcal{R}}$ and characterized by the average occupation numbers $\langle n_j\rangle \equiv \mathrm{Tr}_{\mathcal{R}}\big[\hat{\rho}^0_{\mathcal{R}}\hat{\mathsf{b}}_j^\dagger\hat{\mathsf{b}}_j\big]$ of the modes. The coupling term between $\mathcal{S}$ and $\mathcal{R}$ is chosen of the form

$$\hat{\mathsf{W}} = \hat{\mathsf{S}}\otimes\hat{\mathsf{R}} \quad \text{with} \quad \hat{\mathsf{S}} \equiv \hbar\hat{\sigma}_x\ ,\ \hat{\mathsf{R}} \equiv \sum_j \big(g_j\hat{\mathsf{b}}_j + g_j^*\hat{\mathsf{b}}_j^\dagger\big) \tag{3}$$

with $g_j \in \mathbb{C}$. For later convenience we recall that one may write $\sigma_x = \sigma_+ + \sigma_-$, where the $\{\sigma_\pm\}$ have a simple action on the eigenstates of σ_z.

i. a) Give the interaction representation of the operators $\hat{\mathsf{S}}$ and $\hat{\mathsf{R}}$. [*Hint*: To obtain $\hat{\mathsf{S}}^{(\mathrm{I})}(t)$, find simple differential equation obeyed by the interaction representations of σ_+ and σ_-]. Check that $\hat{\mathsf{R}}^{(\mathrm{I})}(t)$ is centered and compute the correlation function

$$\mathcal{C}(\tau) = \mathrm{Tr}_{\mathcal{R}}\big[\hat{\rho}^0_{\mathcal{R}}\hat{\mathsf{R}}^{(\mathrm{I})}(t+\tau)\hat{\mathsf{R}}^{(\mathrm{I})}(t)\big]. \tag{4}$$

Show that the latter obeys $[\mathcal{C}(\tau)]^* = \mathcal{C}(-\tau)$.

b) Express the two operators (introduced in the lecture)

$$\hat{\mathsf{F}} \equiv \int_0^\infty \mathcal{C}(\tau)\,\hat{\mathsf{S}}^{(\mathrm{I})}(-\tau)\,\mathrm{d}\tau \quad , \quad \hat{\mathsf{G}} \equiv \int_0^\infty \mathcal{C}(-\tau)\,\hat{\mathsf{S}}^{(\mathrm{I})}(-\tau)\,\mathrm{d}\tau \tag{5}$$

in terms of σ_+, σ_- and the two complex numbers

$$c_\pm(\Omega) \equiv \int_0^\infty \mathcal{C}(\tau)\,\mathrm{e}^{\pm\mathrm{i}\Omega\tau}\,\mathrm{d}\tau \tag{6}$$

(and their complex conjugates). You need not compute the $c_\pm(\Omega)$ explicitly yet.

ii. a) Show that the master equation for the reduced statistical operator $\hat{\rho}_{\mathcal{S}}(t)$ describing $\mathcal{S}$ in Schrödinger picture is [for brevity, we denote $c_\pm \equiv c_\pm(\Omega)$]

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}_{\mathcal{S}}(t)}{\mathrm{d}t} = \frac{\mathrm{i}\Omega}{2}\big[\hat{\sigma}_z,\hat{\rho}_{\mathcal{S}}(t)\big] &+ (c_+ + c_+^*)\hat{\sigma}_+\hat{\rho}_{\mathcal{S}}(t)\hat{\sigma}_- - c_+\hat{\sigma}_-\hat{\sigma}_+\hat{\rho}_{\mathcal{S}}(t) - c_+^*\hat{\rho}_{\mathcal{S}}(t)\hat{\sigma}_-\hat{\sigma}_+ \\ &+ (c_- + c_-^*)\hat{\sigma}_-\hat{\rho}_{\mathcal{S}}(t)\hat{\sigma}_+ - c_-\hat{\sigma}_+\hat{\sigma}_-\hat{\rho}_{\mathcal{S}}(t) - c_-^*\hat{\rho}_{\mathcal{S}}(t)\hat{\sigma}_+\hat{\sigma}_- \\ &+ (c_+ + c_-^*)\hat{\sigma}_+\hat{\rho}_{\mathcal{S}}(t)\hat{\sigma}_+ + (c_- + c_+^*)\hat{\sigma}_-\hat{\rho}_{\mathcal{S}}(t)\hat{\sigma}_-. \end{aligned} \tag{7}$$

Hint: You may take the expression of the master equation given in the lecture as starting point.

b) Going momentarily back to the interaction picture, show that the terms in the third line of Eq. (7) oscillate at a higher frequency than those in the first two lines. The "secular approximation" consists in neglecting such rapidly oscillating terms, which we shall do in questions **c)** and **d)**.

c) Inspiring yourself from what was done in the lecture, compute the complex numbers (6), writing them in the form

$$c_\pm(\Omega) = \frac{\Gamma_\pm}{2} - \mathrm{i}\Delta_\pm \tag{8}$$

where the definition of $\Gamma_\pm$ involves a δ-distribution and that of $\Delta_\pm$ a Cauchy principal value.

d) Show that the master equation (7) in the secular approximation becomes

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}_{\mathcal{S}}(t)}{\mathrm{d}t} = \frac{\mathrm{i}(\Omega+\Delta)}{2}\big[\hat{\sigma}_z, \hat{\rho}_{\mathcal{S}}(t)\big] &+ \frac{\Gamma_+}{2}\big[2\hat{\sigma}_+\hat{\rho}_{\mathcal{S}}(t)\hat{\sigma}_- - \big\{\hat{\sigma}_-\hat{\sigma}_+, \hat{\rho}_{\mathcal{S}}(t)\big\}\big] \\ &+ \frac{\Gamma_-}{2}\big[2\hat{\sigma}_-\hat{\rho}_{\mathcal{S}}(t)\hat{\sigma}_+ - \big\{\hat{\sigma}_+\hat{\sigma}_-, \hat{\rho}_{\mathcal{S}}(t)\big\}\big]\end{aligned} \tag{9}$$

with $\Delta \equiv \Delta_- - \Delta_+$. Check that this equation is of the Lindblad form and give the corresponding jump operators $\hat{\mathsf{L}}_\alpha$.

e) If you still have time, deduce from Eq. (7) the evolution equations governing the populations and coherences of $\hat{\rho}_{\mathcal{S}}(t)$. What do you notice regarding the term of Eq. (7) which is neglected in the secular approximation?