Tutorial sheet 7

7. Two-level system coupled to an environment

Consider a two-level system $\mathcal{S}$, whose energy eigenstates will be denoted $|0\rangle$ and $|1\rangle$, coupled to a larger "environment" $\mathcal{R}$ with energy eigenstates $\{|\phi_n^{\mathcal{R}}\rangle\}$. The total system $\mathcal{S}+\mathcal{R}$ is assumed to be at some initial time t_0 in a factorized state

$$|\Psi(t_0)\rangle = |\psi^{\mathcal{S}}\rangle \otimes |\chi^{\mathcal{R}}\rangle.$$

At a later time, the total system will in general no longer be in a factorized state, but in a state of the form

$$|\Psi(t)\rangle = c_0|0\rangle \otimes |\chi_0^{\mathcal{R}}(t)\rangle + c_1|1\rangle \otimes |\chi_1^{\mathcal{R}}(t)\rangle. \tag{1}$$

with $c_0, c_1 \neq 0$, and where $|\chi_0^{\mathcal{R}}(t)\rangle$ and $|\chi_1^{\mathcal{R}}(t)\rangle$ are not necessarily orthogonal (but still normalized to unity). The dependence on the variable t will be dropped from the expressions in the following.

i. Show that the statistical operator corresponding to the pure state (1) is given by

$$\hat{\rho} = |c_0|^2 |0\rangle\langle 0| \otimes |\chi_0^{\mathcal{R}}\rangle\langle\chi_0^{\mathcal{R}}| + |c_1|^2 |1\rangle\langle 1| \otimes |\chi_1^{\mathcal{R}}\rangle\langle\chi_1^{\mathcal{R}}| + c_0c_1^* |0\rangle\langle 1| \otimes |\chi_0^{\mathcal{R}}\rangle\langle\chi_1^{\mathcal{R}}| + c_0^*c_1 |1\rangle\langle 0| \otimes |\chi_1^{\mathcal{R}}\rangle\langle\chi_0^{\mathcal{R}}|. \tag{2}$$

ii. Tracing over the degrees of freedom of the environment (*Hint*: use the basis $\{|\phi_n^{\mathcal{R}}\rangle\}$), show that the reduced statistical operator describing the system $\mathcal{S}$ is given by

$$\hat{\rho}^{\mathcal{S}} = \mathrm{Tr}_{\mathcal{R}}(\hat{\rho}) = \begin{pmatrix} |c_0|^2 & c_0c_1^*\langle\chi_1^{\mathcal{R}}|\chi_0^{\mathcal{R}}\rangle \\ c_0^*c_1\langle\chi_0^{\mathcal{R}}|\chi_1^{\mathcal{R}}\rangle & |c_1|^2 \end{pmatrix} \tag{3}$$

in the basis $\{|0\rangle, |1\rangle\}$. Discuss qualitatively the evolution of $\hat{\rho}^{\mathcal{S}}$ with time.