Tutorial sheet 5

5. Wigner distribution of the one-dimensional harmonic oscillator (2)

i. Preliminary: Phase space trajectory of a classical harmonic oscillator

Consider a classical one-dimensional harmonic oscillator with the Hamilton function

$$H(x,p) = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2. \tag{1}$$

Give the time-dependent solutions $x(t)$, $p(t)$ for a system described by Eq. (1) with the initial conditions $x(t=t_0) = x_0$, $p(t=t_0) = p_0$ at some time t_0. Express (x_0, p_0) as a function of $x(t)$ and $p(t)$.

ii. Quantum harmonic oscillator

a) The Wigner distribution for a particle without spin propagating in one dimension in an arbitrary potential $V(x)$ satisfies the evolution equation

$$\frac{\partial \rho_W(t,x,p)}{\partial t} = -\frac{p}{m}\frac{\partial \rho_W(t,x,p)}{\partial x} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(\frac{\mathrm{i}\hbar}{2}\right)^{2n} \frac{\partial^{2n+1}V(x)}{\partial x^{2n+1}} \frac{\partial^{2n+1}\rho_W(t,x,p)}{\partial p^{2n+1}}. \tag{2}$$

How does this equation simplify for a one-dimensional (quantum) harmonic oscillator?

b) Instead of solving the equation you obtained in the previous question, one can notice that it coincides with the Liouville equation describing the motion of the phase-space probability distribution of the *classical* harmonic oscillator. Inspiring yourself from the functions $x_0 = x_0\big(x(t), p(t)\big)$ and $p_0 = p_0\big(x(t), p(t)\big)$ determined in question **i.**, give a relation between $\rho_\mathrm{W}(t,x,p)$ and ρ_W evaluated at an initial time t_0 at another phase space point. Check that this solution indeed fulfills the evolution equation for ρ_W.

Hint: The solution $\rho_\mathrm{W}(t,x,p)$ is given on the next-to-next page. But only look at it after searching alone for some time!

c) Assume that the initial Wigner density is given by

$$\rho_\mathrm{W}(t_0,x,p) = \frac{1}{\pi\hbar}\mathrm{e}^{-\xi p^2/\hbar m\omega}\mathrm{e}^{-m\omega(x-\bar{x}_0)^2/\hbar\xi}, \tag{3}$$

where $\bar{x}_0 \in \mathbb{R}$ and $\xi > 0$. If $\xi = 1$, this coincides with the Wigner function for the ground state of the harmonic oscillator [exercise **4.i.**], shifted by $\bar{x}_0$ in position space.

Using $\hbar = 1$, $m\omega = 1$, and $\bar{x}_0 = 5$, plot (3D) $\rho_\mathrm{W}(t,x,p)$ at successive times: t_0, $t_0 + \pi/2\omega$, $t_0 + \pi/\omega$, $t_0 + 3\pi/2\omega$ [some among you might even be able to prepare a movie!] for the cases $\xi = 1$ and $\xi = 3$.

Check that for $\xi = 1$ the position and momentum dependencies of $\rho_\mathrm{W}(t,x,p)$ factorize. What does this mean for the time evolution of the position probability distribution? Plot (2D) the latter at successive times as above.

Solution to **ii.b)**:

$$\rho_{\mathrm{W}}(t,x,p) = \rho_{\mathrm{W}}\left(t_0, x\cos[\omega(t-t_0)] - \frac{p}{m\omega}\sin[\omega(t-t_0)], p\cos[\omega(t-t_0)] + m\omega x\sin[\omega(t-t_0)]\right)$$