i. Phase space trajectory of a classical harmonic oscillator

Starting from the Hamilton function

$$H(x,p) = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2. \tag{1}$$

one deduces the Hamilton equations

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad , \quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x \tag{2}$$

with the (known) solution

$$x(t) = x_0\cos[\omega(t-t_0)] + \frac{p_0}{m\omega}\sin[\omega(t-t_0)] \quad , \quad p(t) = p_0\cos[\omega(t-t_0)] - m\omega x_0\sin[\omega(t-t_0)] \tag{3}$$

which corresponds to an ellipse in phase space. Inverting these relations, one can write

$$x_0 = x(t)\cos[\omega(t-t_0)] - \frac{p(t)}{m\omega}\sin[\omega(t-t_0)] \quad , \quad p_0 = p(t)\cos[\omega(t-t_0)] + m\omega x(t)\sin[\omega(t-t_0)]. \tag{4}$$

ii. Quantum harmonic oscillator

a) The only nonvanishing odd derivative of the potential $V(x) = \frac{1}{2}m\omega^2x^2$ is the first one, so that the evolution equation for the Wigner distribution of the quantum harmonic oscillator becomes (keeping only the term $n=0$ in the sum)

$$\frac{\partial\rho_W(t,x,p)}{\partial t} = -\frac{p}{m}\frac{\partial\rho_W(t,x,p)}{\partial x} + m\omega^2x\frac{\partial\rho_W(t,x,p)}{\partial p}. \tag{5}$$

b) In the case of the classical harmonic oscillator, a system which is at the phase-space point $\big(x(t),p(t)\big)$ at time t was (or will be!) at (x_0,p_0) given by relation (4) at time t_0. This suggests the following ansatz for the Wigner distribution of the quantum harmonic oscillator:

$$\rho_{\rm W}(t,x,p) = \rho_{\rm W}\bigg(t_0, x\cos[\omega(t-t_0)] - \frac{p}{m\omega}\sin[\omega(t-t_0)], p\cos[\omega(t-t_0)] + m\omega x\sin[\omega(t-t_0)]\bigg), \tag{6}$$

with the interpretation that what is found at (x,p) at time t corresponds to what is at $\big(x_0(x,p),p_0(x,p)\big)$ at time t_0.

One can indeed compute the derivatives of the r.h.s. of Eq. (6) with respect to t

$$\frac{\partial\rho_W(t,x,p)}{\partial t} = -\bigg(\omega x\sin[\omega(t-t_0)] + \frac{p}{m}\cos[\omega(t-t_0)]\bigg)\frac{\partial\rho_W\big|_0}{\partial x} + \Big(-\omega p\sin[\omega(t-t_0)] + m\omega^2x\cos[\omega(t-t_0)]\Big)\frac{\partial\rho_W\big|_0}{\partial p},$$

[where the notation $\big|_0$ stands for the arguments on the r.h.s. of Eq. (6)], to x

$$\frac{\partial\rho_W(t,x,p)}{\partial x} = \cos[\omega(t-t_0)]\frac{\partial\rho_W\big|_0}{\partial x} + m\omega\sin[\omega(t-t_0)]\frac{\partial\rho_W\big|_0}{\partial p},$$

and eventually with respect to p

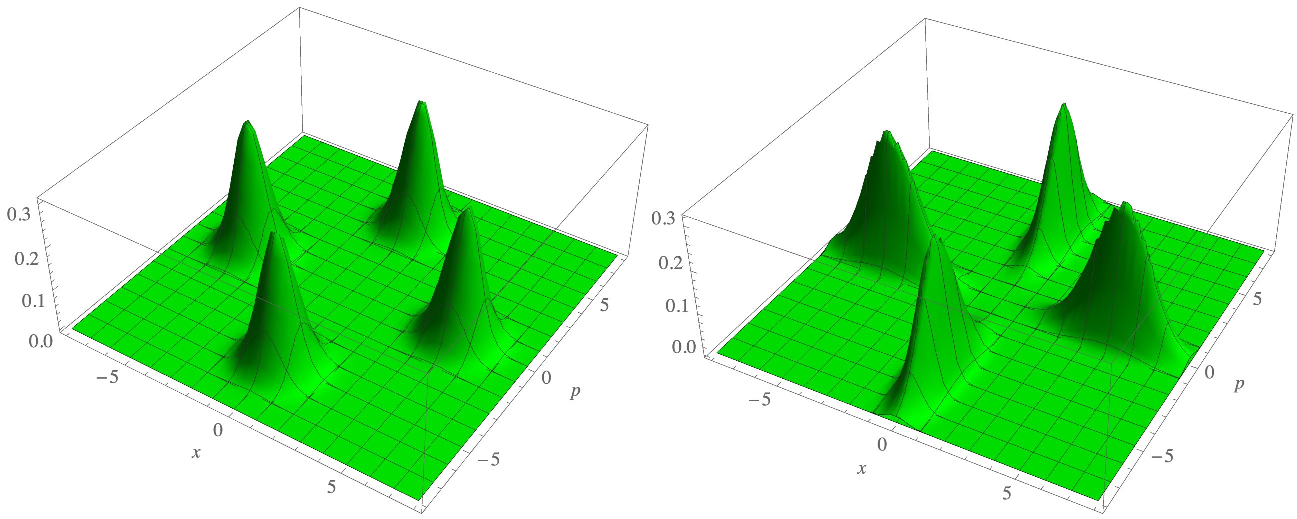
$$\frac{\partial\rho_W(t,x,p)}{\partial p} = -\frac{\sin[\omega(t-t_0)]}{m\omega}\frac{\partial\rho_W\big|_0}{\partial x} + \cos[\omega(t-t_0)]\frac{\partial\rho_W\big|_0}{\partial p},$$

from which one sees that these derivatives satisfy Eq. (5).

c) We consider the Wigner distribution

$$\rho_{\rm W}(t,x,p) = \frac{1}{\pi\hbar}\exp\left[-\frac{\xi}{\hbar m\omega}\Big(p\cos[\omega(t-t_0)] + m\omega x\sin[\omega(t-t_0)]\Big)^2 - \frac{m\omega}{\hbar\xi}\Big(x\cos[\omega(t-t_0)] - \frac{p}{m\omega}\sin[\omega(t-t_0)] - \bar{x}_0\Big)^2\right] \tag{7}$$

for various values of $\xi > 0$. The profiles of the Wigner distributions with $\xi = 1$ (left) and $\xi = 3$ (right) at the four successive times $t = t_0 + n\pi/2\omega, n \in \{0, 1, 2, 3\}$ are shown below.



In both plots, the distribution at time t_0 is that centered about ($\bar{x}_0 = 5$, $p = 0$), and the state moves clockwise about the phase-space origin (see the notebook for a "movie" of the time evolution).

In the case $\xi = 1$, one finds that the distribution (7) can be rewritten as

$$\rho_{\rm W}(t,x,p) = \frac{1}{\pi\hbar}\exp\left[-\frac{1}{\hbar m\omega}\Big(p + m\omega\bar{x}_0\sin[\omega(t-t_0)]\Big)^2 - \frac{m\omega}{\hbar}\Big(x - \bar{x}_0\cos[\omega(t-t_0)]\Big)^2\right], \tag{8}$$

i.e. as the product of a Gaussian in p times another Gaussian in x. The integral over momentum space is thus straightforward and yields the position probability distribution

$$p(t,x) \equiv \int \rho_{\rm W}(t,x,p)\,{\rm d}p = \sqrt{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{\hbar}\Big(x - \bar{x}_0\cos[\omega(t-t_0)]\Big)^2\right]. \tag{9}$$

One sees at once that this probability is a Gaussian distribution with constant width $\sqrt{\hbar/2m\omega}$ centered about the oscillating average value $\bar{x}_0\cos[\omega(t-t_0)]$.

In contrast, for $\xi = 3$ the position probability distribution oscillates between $-\bar{x}_0$ and $\bar{x}_0$ with a time-dependent width (and height).