Tutorial sheet 4

4. Wigner distribution of the one-dimensional harmonic oscillator

Consider a particle without spin propagating in one dimension in a quadratic potential, described by the Hamilton operator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2. \tag{1}$$

i. Compute the Wigner distribution for the case where the particle is **a)** in the ground state; **b)** in the first excited state. The corresponding wave functions in position representation are

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \mathrm{e}^{-m\omega x^2/2\hbar} \quad , \quad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}}\, x\, \mathrm{e}^{-m\omega x^2/2\hbar}. \tag{2}$$

Check that the respective values of $\rho_{\mathrm{W}}^{(n)}$ (here with $n = 0$ or 1) at $(x, p) = (0, 0)$ match those that you calculated (actually, in the 3D case) in the lecture on May 5 for wave functions with a given parity. Plot $\rho_{\mathrm{W}}^{(0)}$ and $\rho_{\mathrm{W}}^{(1)}$ (3-dimensional plots), using the values $\hbar = 1$ and $m\omega = 1$.

ii. Using the (simple!) Wigner transform H_{W} of the Hamiltonian (1) and the Wigner distribution $\rho_{\mathrm{W}}^{(0)}$, compute the expectation value of the energy in the ground state. Discuss (critically!) your result.

iii. One can show (you may try to prove it) that the Wigner transform of $\hat{H}^2$ for the Hamiltonian (1) is

$$\left(H^2\right)_{\mathrm{W}}(x, p) = \left[H_{\mathrm{W}}(x, p)\right]^2 - \frac{(\hbar\omega)^2}{4}. \tag{3}$$

Compute (in Wigner representation) the expectation value of $\hat{H}^2$ in the ground state. Compare with the square of the expectation value you found in **iii.** and discuss.