Tutorial sheet 3

Discussion topic: What is the Berry phase?

3. Berry curvature

For a system with a Hamilton operator $\hat{H}[\boldsymbol{R}(t)]$ depending on $\mathscr{D}$ time-dependent parameters $\{R_i(t)\}$ collectively denoted $\boldsymbol{R}(r)$, the Berry phase $\bar{\gamma}_n$ for the n-th eigenstate when the system adiabatically describes a closed path $\mathcal{C}$ in parameter space reads

$$\bar{\gamma}_n = \mathrm{i}\oint_{\mathcal{C}} \big\langle \psi_n(\boldsymbol{R}) \big| \boldsymbol{\nabla}_{\boldsymbol{R}} \psi_n(\boldsymbol{R}) \big\rangle \cdot \mathrm{d}\boldsymbol{R} \quad \text{with} \quad \boldsymbol{\mathcal{A}}_n(\boldsymbol{R}) \equiv \mathrm{i}\big\langle \psi_n(\boldsymbol{R}) \big| \boldsymbol{\nabla}_{\boldsymbol{R}} \psi_n(\boldsymbol{R}) \big\rangle, \tag{1}$$

where $|\psi_n(\boldsymbol{R})\rangle$ denotes a normalized eigenvector associated with the n-th eigenvalue $E_n(\boldsymbol{R})$ of $\hat{H}[\boldsymbol{R}]$. Invoking Stokes' theorem, the line integral of the Berry connection can be replaced by an integral over a surface $\mathcal{S}$ bounded by $\mathcal{C}$ of the Berry curvature, defined as a $\mathscr{D}$-dimensional antisymmetric rank-2 tensor $\boldsymbol{\mathcal{B}}_n(\boldsymbol{R})$ with components

$$\boldsymbol{\mathcal{B}}_n^{jk}(\boldsymbol{R}) \equiv \frac{\partial}{\partial R_j}\boldsymbol{\mathcal{A}}_n^k(\boldsymbol{R}) - \frac{\partial}{\partial R_k}\boldsymbol{\mathcal{A}}_n^j(\boldsymbol{R}). \tag{2}$$

i. Check that the components (2) of the Berry curvature can be expressed as

$$\boldsymbol{\mathcal{B}}_n^{jk}(\boldsymbol{R}) = \mathrm{i}\left[\left\langle \frac{\partial\psi_n(\boldsymbol{R})}{\partial R_j}\middle|\frac{\partial\psi_n(\boldsymbol{R})}{\partial R_k}\right\rangle - \left\langle \frac{\partial\psi_n(\boldsymbol{R})}{\partial R_k}\middle|\frac{\partial\psi_n(\boldsymbol{R})}{\partial R_j}\right\rangle\right]. \tag{3}$$

What does this give in case the eigenvector $\psi_n(\boldsymbol{R})$ is real?

ii. Show that the component (4) can be rewritten as

$$\boldsymbol{\mathcal{B}}_n^{jk}(\boldsymbol{R}) = \mathrm{i}\sum_{n'\neq n} \frac{\big\langle \psi_n(\boldsymbol{R})\big|\partial\hat{H}/\partial R_j\big|\psi_{n'}(\boldsymbol{R})\big\rangle\big\langle \psi_{n'}(\boldsymbol{R})\big|\partial\hat{H}/\partial R_k\big|\psi_n(\boldsymbol{R})\big\rangle - (j \leftrightarrow k)}{\big[E_n(\boldsymbol{R}) - E_{n'}(\boldsymbol{R})\big]^2}, \tag{4}$$

where the second term denoted $(j \leftrightarrow k)$ in the numerator follows from the first term after exchanging the indices j and k. How does the Berry curvature change if the Hamilton operator is multiplied by a real number $\lambda \neq 0$?

iii. What is the "total Berry curvature" $\sum_n \boldsymbol{\mathcal{B}}_n^{jk}(\boldsymbol{R})$ of the system?

Hint: You may first try with a 2-level system to get an idea of the result.