Tutorial sheet 2

Discussion topic: What is the physics idea underlying the adiabatic approximation? What is the "adiabatic theorem"?

2. Forced quantum harmonic oscillator

Consider a one-dimensional harmonic oscillator with mass m and characteristic angular frequency ω, acted upon by an "external" time-dependent force, such that the overall Hamilton operator of the system reads (in position representation)

$$\hat{H}(t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2 - m\omega^2 x f(t), \tag{1}$$

where it will be assumed that $f(t{=}0) = 0$. Note that $f(t)$ has the dimension of a length.

i. Refresh your knowledge of the free harmonic oscillator, in particular its energy levels and the operation of the ladder operators $\hat{a}$ and $\hat{a}^\dagger$ on the energy eigenstates $|n\rangle$ with $n \in \mathbb{N}$.
To fix notations, let $\phi_n(x) \equiv \langle x|n\rangle$ denote the corresponding eigenfunctions in position representation, whose explicit form is not needed in this exercise.

ii. Energy eigenvalues and eigenstates

Give the "instantaneous" eigenvalues $\{E_n(t)\}$ and eigenfunctions $\{\psi_n(x,t)\}$ (expressed with the help of the functions $\{\phi_n(x)\}$) of the Hamilton operator (1).

Hint: No complicated calculation is needed, only some appropriate rewriting of $\hat{H}(t)$.

iii. Adiabatic approximation

a) State the precise criterion — expressed as a condition on f and/or its time derivative $\dot{f}$ — for the adiabaticity of the evolution. Show (*hint*: using an integration by parts) that in that regime the quantity

$$x_{\text{cl.}}(t) \equiv \int_0^t f(t')\omega \sin[\omega(t-t')]\,\mathrm{d}t' \tag{2}$$

is approximately equal to $f(t)$.

b) Check that $x_{\text{cl.}}(t)$ is a solution of the *classical* equation of motion for the forced harmonic oscillator with appropriate initial conditions at time $t = 0$.

iv. Beyond the adiabatic approximation

Writing[1] $\Psi(x,t) = \sum_n c_n(t)\psi_n(x,t)$ for the state of the oscillator at time t, the time evolution of the coefficients $c_n(t)$ is given by Eq. [10.19] in Griffiths. Using the instantaneous eigenstates and eigenvalues determined in **ii.** and properties recalled in **i.**, show that the infinite sum entering this evolution equation can actually be simplified significantly.

Hint: Express the position operator $\hat{x}$ in terms of the ladder operators.

[1] I use the same notations as in Griffiths' *Introduction to Quantum Mechanics*.