Tutorial sheet 10

9. A model for depolarization

Consider a two-level system $\mathcal{S}$ coupled with a four-level environment $\mathcal{R}$, whose states can be decomposed on an orthonormal basis $\{|r_\mu\rangle\}$ where $\mu \in \{0,1,2,3\}$. When $\mathcal{S}$ is initially in a state $|\Psi_{\mathcal{S}}\rangle$ and the environment in the state $|r_0\rangle$, the system $\mathcal{S}+\mathcal{R}$ can evolve by according to the law

$$|\Psi_{\mathcal{S}}\rangle \otimes |r_0\rangle \longrightarrow \sqrt{1-p}\,|\Psi_{\mathcal{S}}\rangle \otimes |r_0\rangle + \sqrt{\frac{p}{3}} \sum_{\mu=1}^{3} \big(\hat{\sigma}_\mu \otimes \hat{\mathbb{1}}_{\mathscr{H}_{\mathcal{R}}}\big)|\Psi_{\mathcal{S}}\rangle \otimes |r_\mu\rangle, \tag{1}$$

where the operator $\hat{\sigma}_\mu$ is the μ-th Pauli matrix acting on the Hilbert space $\mathscr{H}_{\mathcal{S}}$ of $\mathcal{S}$.

Show that a possible choice of Kraus operators to represent this evolution is

$$\hat{\mathsf{E}}_0 = \sqrt{1-p}\,\hat{\mathbb{1}}_{\mathscr{H}_{\mathcal{S}}} \quad , \quad \hat{\mathsf{E}}_\mu = \sqrt{\frac{p}{3}}\,\hat{\sigma}_\mu \tag{2}$$

Check that they satisfy the completeness relation and compute the superoperator $\mathcal{E}\big(\hat{\rho}_{\mathcal{S}}\big)$ corresponding to the evolution law (1). Show that if $\hat{\rho}_{\mathcal{S}}$ is written in the form (why is this always possible?)

$$\hat{\rho}_{\mathcal{S}} = \frac{1}{2}\big(\hat{\mathbb{1}}_{\mathscr{H}_{\mathcal{S}}} + \vec{b}\cdot\hat{\vec{\sigma}}\big) \tag{3}$$

with $\vec{b} \in \mathbb{R}^3$ and $|\vec{b}| = 1$,[1] then

$$\mathcal{E}\big(\hat{\rho}_{\mathcal{S}}\big) = \frac{1}{2}\big(\hat{\mathbb{1}}_{\mathscr{H}_{\mathcal{S}}} + \vec{b}'\cdot\hat{\vec{\sigma}}\big) \quad \text{with} \quad \vec{b}' = \left(1 - \frac{4p}{3}\right)\vec{b}. \tag{4}$$

Why is the exercise titled as it is?

[1] That is, $\vec{b}$ belongs to the unit sphere S^2, which in this context is called *Bloch sphere*.