Tutorial sheet 9

Discussion topic: Langevin model of Brownian motion

22. Examples of Markov processes

The lecture introduced the so-called *Markov processes*, which are entirely determined by their single-time density $p_{Y,1}$ and their conditional probability density $p_{Y,1|1}$. The latter, which is referred to as *transition probability*, obeys the *Chapman–Kolmogorov equation*

$$p_{Y,1|1}(t_3, y_3 \,|\, t_1, y_1) = \int p_{Y,1|1}(t_3, y_3 \,|\, t_2, y_2)\, p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)\, \mathrm{d}y_2 \quad \text{for } t_1 < t_2 < t_3, \tag{1}$$

to be compared with Eq. (1) of exercise **20**.

i. Wiener process

The stochastic process defined by the "initial condition" $p_{Y,1}(t{=}0, y) = \delta(y)$ for $y \in \mathbb{R}$ and the transition probability $(0 < t_1 < t_2)$

$$p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1) = \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp\left[-\frac{(y_2 - y_1)^2}{2(t_2 - t_1)}\right]$$

is called *Wiener process*.

Check that this transition probability obeys the Chapman–Kolmogorov equation, and that the probability density at time $t > 0$ is given by

$$p_{Y,1}(t, y) = \frac{1}{\sqrt{2\pi t}}\, \mathrm{e}^{-y^2/2t}.$$

Remark: Note that the above single-time probability density is solution of the diffusion equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}$$

with diffusion coefficient $D = \frac{1}{2}$.

ii. Ornstein–Uhlenbeck process

The so-called *Ornstein–Uhlenbeck process* is defined by the time-independent single-time probability density

$$p_{Y,1}(y) = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-y^2/2}$$

and the transition probability $(\tau > 0)$

$$p_{Y,1|1}(t + \tau, y \,|\, t, y_0) = \frac{1}{\sqrt{2\pi(1 - \mathrm{e}^{-2\tau})}} \exp\left[-\frac{(y - y_0\mathrm{e}^{-\tau})^2}{2(1 - \mathrm{e}^{-2\tau})}\right].$$

a) Check that this transition probability fulfills the Chapman–Kolmogorov equation, so that the Ornstein–Uhlenbeck process is Markovian. Show that the process is also Gaussian, stationary, and that its autocorrelation function is $\kappa(\tau) = \mathrm{e}^{-\tau}$.

b) What is the large-τ limit of the transition probability? And its limit when τ goes to 0^+?

c) Viewing the above transition probability as a function of τ and y, can you find a partial differential equation, of which it is a (fundamental) solution?

Hint: Let yourself be inspired(?) by the remark at the end of question **i.**

23. Vibrating string

Consider a weightless elastic string, whose extremities are fixed at points $x = 0$ and $x = L$ along the x-axis. Let $y(x)$ denote the displacement of the string transverse to this axis—for the sake of simplicity, we can assume that this displacement is one-dimensional—at position x. For small displacements, one can show that the elastic energy associated with a given profile $y(x)$ reads

$$E[y(x)] = \int_0^L \frac{1}{2} k \left[\frac{\mathrm{d}y(x)}{\mathrm{d}x}\right]^2 \mathrm{d}x, \tag{2}$$

with k a positive constant.

When the string undergoes thermal fluctuations, induced by its environment at temperature T, $y(x)$ becomes a random function (of position, instead of time), where one expects that the probability for a given $y(x)$ should be proportional to $\mathrm{e}^{-\beta E[y(x)]}$ with $\beta = 1/k_B T$. Here, we wish to consider a discretized version of the problem and view $y(x)$ as the realization of a stochastic function $Y(x)$.

i. Let $n \in \mathbb{N}$. Consider n points $0 < x_1 < x_2 < \cdots < x_n < L$ and let y_j be the displacement of the string at point x_j. Write down the energy of the string, assuming that it is straight between two successive points x_j, x_{j+1}.

Hint: For the sake of brevity, one can introduce the notations $x_0 = 0$, $x_{n+1} = L$, $y_0 = y_{n+1} = 0$.

ii. We introduce the n-point probability density

$$p_n(x_1, y_1; \ldots; x_n, y_n) = \sqrt{\frac{2\pi L}{k\beta}} \prod_{j=0}^{n} \sqrt{\frac{k\beta}{2\pi(x_{j+1} - x_j)}} \exp\left[-\frac{k\beta}{2} \frac{(y_{j+1} - y_j)^2}{x_{j+1} - x_j}\right],$$

which for large n, agrees with the anticipated factor $\mathrm{e}^{-\beta E[y(x)]}$ (are you convinced of that?).

Show that the various p_n satisfy the 4 properties of n-point densities given in the lecture. Write down the single-point and two-point averages $\langle Y(x_1)\rangle$ and $\langle Y(x_1)Y(x_2)\rangle$, as well as the autocorrelation function. Which properties does the process possess?