Tutorial sheet 8

19. Electrical conductivity in a magnetic field: Hall effect

This exercise is a sequel to exercise **17**, yet tackles the problem differently. We again consider the problem of electric conduction in a metal. We now assume that the conductor subject to the electric and magnetic fields $\vec{\mathscr{E}}$, $\vec{\mathscr{B}}$ is a rectangular parallelepiped, with its sides along the coordinate axes. Let L, l, and d be the respective lengths of the sides parallel to the x-, y and z-directions.

i. Drude–Lorentz model

To model the effect of collisions in a simple way, one introduces an "average equation of motion" for the conduction electrons—i.e., an evolution equation for their average velocity $\langle\vec{v}\rangle$

$$\frac{\mathrm{d}\langle\vec{v}\rangle}{\mathrm{d}t} = -\frac{\langle\vec{v}\rangle}{\tau_{\mathrm{r}}} - \frac{e}{m_{\mathrm{e}}}\big(\vec{\mathscr{E}} + \langle\vec{v}\rangle \times \vec{\mathscr{B}}\big).$$

Give a physical interpretation for this equation. Check that in the stationary regime one has

$$\langle v_x\rangle = -\frac{e\tau_{\mathrm{r}}}{m_{\mathrm{e}}}\,\mathscr{E}_x - \omega\tau_{\mathrm{r}}\langle v_y\rangle\,, \qquad \langle v_y\rangle = -\frac{e\tau_{\mathrm{r}}}{m_{\mathrm{e}}}\,\mathscr{E}_y + \omega\tau_{\mathrm{r}}\langle v_x\rangle\,,$$

with ω the Larmor frequency defined in exercise **17**. Show that if one takes $\tau_{\mathrm{r}} = \tau_{\mathrm{F}}$, one recovers the same expression for the conductivity tensor as in exercise **17**.

ii. Calculate in terms of $\mathscr{E}_x$ the value $\mathscr{E}_{\mathrm{H}}$ of $\mathscr{E}_y$ which cancels $(J_{\mathrm{el.}})_y$. Verify that the transport of electrons in that situation is the same as in the case $\vec{\mathscr{B}} = \vec{0}$, in other words $(J_{\mathrm{el.}})_x = \sigma_{\mathrm{el.}}\mathscr{E}_x$. The field intensity $\mathscr{E}_{\mathrm{H}}$ is called *Hall field*, and the *Hall resistance* is defined by

$$R_{\mathrm{H}} \equiv \frac{V_{\mathrm{H}}}{I}$$

where V_{H} is the *Hall voltage*, $V_{\mathrm{H}}/l = \mathscr{E}_{\mathrm{H}}$, and I the total electric current along the x direction. Show that R_{H} is given by

$$R_{\mathrm{H}} = \frac{\mathscr{B}}{nde},$$

with n the density of conduction electrons and $\mathscr{B} \equiv |\vec{\mathscr{B}}|$. By noting that R_{H} is independent of the relaxation time, find its expression using an elementary argument.

20. Stochastic processes

Let $Y(t)$ be a stochastic process and $p_{Y,n}$ resp. $p_{Y,n|m}$ its n-point resp. conditional n-point densities.

i. Starting from the expression of Bayes' theorem, show that for all integers $n \geq 2$ one can write

$$p_{Y,n}(t_1,y_1;\dots;t_n,y_n) = p_{Y,1|n-1}(t_n,y_n\,|\,t_1,y_1;\dots;t_{n-1},y_{n-1}) \times p_{Y,1|n-2}(t_{n-1},y_{n-1}\,|\,t_1,y_1;\dots;t_{n-2},y_{n-2})\cdots p_{Y,1|1}(t_2,y_2\,|\,t_1,y_1)\,p_{Y,1}(t_1,y_1),$$

where the n instants $t_1, t_2, \dots, t_n$ are all different, but otherwise arbitrary.

ii. Consider the previous identity for $n = 3$. Using the consistency condition which expresses $p_{Y,m}$ as an integral of $p_{Y,n}$ with $n > m$, here with $m = 2$, show that the single-time conditional density $p_{Y,1|1}$ obeys the integral-functional equation

$$p_{Y,1|1}(t_3,y_3\,|\,t_1,y_1) = \int p_{Y,1|2}(t_3,y_3\,|\,t_1,y_1;t_2,y_2)\,p_{Y,1|1}(t_2,y_2\,|\,t_1,y_1)\,\mathrm{d}y_2. \qquad (1)$$

Generalize this relation to an equation for $p_{Y,1|n}$ involving $p_{Y,1|n+1}$.

21. Characteristic functional of a stochastic process

In the lecture, the characteristic functional associated with a stochastic process $Y_X(t)$ has been defined as

$$G_Y[k(t)] \equiv \left\langle \exp\left[\mathrm{i}\int k(t)\,Y(t)\,\mathrm{d}t\right]\right\rangle,$$

with $k(t)$ a test function.

Expand the exponential in power series of k and express the characteristic functional in terms of the n-time moments. How would you write the moment $\langle Y(t_1)Y(t_2)\cdots Y(t_n)\rangle$ as function of $G_Y[k(t)]$?