Tutorial sheet 7

Discussion topic: Boltzmann equation: relaxation-time approximation

17. Electrical conductivity in a magnetic field[1]

We consider the problem of electric conduction in a metal subject to a constant and uniform electromagnetic field $(\vec{\mathscr{E}},\vec{\mathscr{B}})$. The conduction electrons (mass $m_{\rm e}$, charge $-e$) form a non-relativistic, highly degenerate ideal Fermi gas obeying the kinetic Lorentz equation (cf. Tutorial sheet 5) in the presence of an external force $\vec{F}$, which is here simply the Lorentz force. We assume that the various densities are uniform and stationary: $\bar{\mathsf{f}}(t,\vec{r},\vec{p}) = \bar{\mathsf{f}}(\vec{p})$, where $\vec{p}$ denotes the linear momentum, and $n(t,\vec{r}) = n$, which leads to simplifications on the left-hand side of the Lorentz equation. In addition, we assume that the local equilibrium distribution is a function of energy only: $\bar{\mathsf{f}}^{(0)}(\vec{p}) = \bar{\mathsf{f}}^{(0)}(\varepsilon)$ with $\varepsilon \equiv \vec{p}^{\,2}/2m_{\rm e}$.

i. We first take $\vec{\mathscr{B}} = \vec{0}$. Calculate $\delta\bar{\mathsf{f}} \equiv \bar{\mathsf{f}} - \bar{\mathsf{f}}^{(0)}$ and show that the electric current density $\vec{J}_{\rm el.}$ is given in the relaxation time approximation by

$$\vec{J}_{\rm el.} = -e^2 \int \tau_{\rm r}(|\vec{p}|)(\vec{v}\cdot\vec{\mathscr{E}})\,\vec{v}\,\frac{{\rm d}\bar{\mathsf{f}}^{(0)}}{{\rm d}\varepsilon}\,\frac{{\rm d}^3\vec{p}}{(2\pi\hbar)^3}.$$

Show that, if the local equilibrium distribution is the Fermi distribution (with 2 spin degrees of freedom) at $T=0$, $\bar{\mathsf{f}}^{(0)}(\varepsilon) = 2\,\Theta(\varepsilon_{\rm F}-\varepsilon)$ with $\varepsilon_{\rm F}$ the Fermi energy, then $\vec{J}_{\rm el.}$ obeys Ohm's law with the electrical conductivity

$$\sigma_{\rm el.} = \frac{n\,e^2}{m_{\rm e}}\tau_{\rm F}$$

where $\tau_{\rm F} \equiv \tau_{\rm r}(p_{\rm F})$, with $p_{\rm F}$ the Fermi momentum.

ii. Let now $\vec{\mathscr{B}} \neq \vec{0}$. How is the electrical conductivity modified if $\vec{\mathscr{B}}$ is parallel to $\vec{\mathscr{E}}$?

iii. Consider the case where the electric field is in the xy-plane and the magnetic field along the z-axis, $\vec{\mathscr{B}} = \mathscr{B}\,\vec{\rm e}_z$ with $\mathscr{B} > 0$.

a) Show that the Lorentz equation in the relaxation time approximation becomes

$$-e\vec{v}\cdot\vec{\mathscr{E}}\,\frac{{\rm d}\bar{\mathsf{f}}^{(0)}}{{\rm d}\varepsilon} - e\big(\vec{v}\times\vec{\mathscr{B}}\big)\cdot\vec{\nabla}_{\vec{p}}\,\delta\bar{\mathsf{f}} = -\frac{\delta\bar{\mathsf{f}}}{\tau_r(|\vec{p}|)}.$$

b) We look for a solution of the form

$$\delta\bar{\mathsf{f}} = -\vec{v}\cdot\vec{C}\,\frac{{\rm d}\bar{\mathsf{f}}^{(0)}}{{\rm d}\varepsilon}$$

with $\vec{C}$ a vector, function of $\vec{\mathscr{E}}$ and $\vec{\mathscr{B}}$ but independent of $\vec{v}$, to be determined. What should $\vec{C}$ be when $\vec{\mathscr{B}} = \vec{0}$? when $\vec{\mathscr{E}} = \vec{0}$? For the latter case, estimate first the average magnetic force on the electrons.

c) Show that $\vec{C}$ satisfies the equation

$$-e\vec{\mathscr{E}} + \vec{\omega}\times\vec{C} = \frac{\vec{C}}{\tau_r(|\vec{p}|)},$$

with $\vec{\omega} = \omega\,\vec{\rm e}_z$, where $\omega \equiv e\mathscr{B}/m_{\rm e}$ is the Larmor frequency of the electrons. Explain why $\vec{C}$ is necessarily of the form $\vec{C} = \alpha\vec{\mathscr{E}} + \delta\vec{\mathscr{B}} + \gamma\vec{\mathscr{B}}\times\vec{\mathscr{E}}$, with α,δ,γ real numbers. Find the expression of $\vec{C}$ and show that

$$\delta\bar{\mathsf{f}} = \frac{e\tau_{\rm r}}{1+\omega^2\tau_{\rm r}^2}\big(\vec{\mathscr{E}} + \tau_{\rm r}\,\vec{\omega}\times\vec{\mathscr{E}}\big)\cdot\vec{v}\,\frac{{\rm d}\bar{\mathsf{f}}^{(0)}}{{\rm d}\varepsilon}.$$

[1]This exercise was shamelessly stolen from the book *Equilibrium and non-equilibrium statistical thermodynamics* by M. Le Bellac *et al.*

d) Calculate the electric current and the components $(\sigma_{\text{el.}})_{ij}$, $i,j=x,y$ of the electrical conductivity tensor. Verify that $(\sigma_{\text{el.}})_{xy} = -(\sigma_{\text{el.}})_{yx}$ and comment on this relation.

18. Boltzmann gas in a harmonic trap

Consider a system of N neutral particles of mass m and let $\bar{\mathsf{f}}(t,\vec{r},\vec{p})$ denote its phase-space density.

i. Balance equation

Let $g(\vec{r},\vec{p})$ denote a dynamical quantity.

a) Denoting by $\mathcal{I}_{\text{coll.}}$ the collision integral of the kinetic Boltzmann equation, show that the latter leads to the generic balance equation

$$\frac{\mathrm{d}\langle g\rangle}{\mathrm{d}t} - \big\langle \vec{v}\cdot\vec{\nabla}_{\vec{r}}\, g\big\rangle - \big\langle \vec{F}\cdot\vec{\nabla}_{\vec{p}}\, g\big\rangle = \big\langle g\, \bar{\mathsf{f}}^{-1}\mathcal{I}_{\text{coll.}}\big\rangle, \tag{1}$$

where the angular brackets denote an average over position and momenta:

$$\langle g\rangle \equiv \frac{1}{N}\int g(\vec{r},\vec{p})\, f(t,\vec{r},\vec{p})\, \mathrm{d}^6\mathcal{V}.$$

Hint: You may use the fact that f vanishes "at the edges" of phase space.

b) Explain why $\big\langle g\, \bar{\mathsf{f}}^{-1}\mathcal{I}_{\text{coll.}}\big\rangle$ vanishes when g is of the form $g(\vec{r},\vec{p}) = a(\vec{r}) + \vec{b}(\vec{r})\cdot\vec{p} + c(\vec{r})\vec{p}^{\,2}$ with a, $\vec{b}$, c arbitrary functions of $\vec{r}$.

ii. Let the system be trapped in a harmonic potential, namely $\vec{F} = -\vec{\nabla}V$ with $V(\vec{r}) = \frac{1}{2}m\omega_0^2\vec{r}^{\,2}$.

a) Derive using Eq. (1) the coupled system of equations

$$\frac{\mathrm{d}\langle\vec{r}^{\,2}\rangle}{\mathrm{d}t} = 2\big\langle\vec{r}\cdot\vec{v}\big\rangle \quad,\quad \frac{\mathrm{d}\langle\vec{r}\cdot\vec{v}\rangle}{\mathrm{d}t} = \big\langle\vec{v}^{\,2}\big\rangle - \omega_0^2\big\langle\vec{r}^{\,2}\big\rangle \quad,\quad \frac{\mathrm{d}\langle\vec{v}^{\,2}\rangle}{\mathrm{d}t} = -2\omega_0^2\big\langle\vec{r}\cdot\vec{v}\big\rangle. \tag{2}$$

b) Find and discuss the solutions of the system (2) evolving in time like $\mathrm{e}^{\mathrm{i}\omega t}$.

The non-trivial behavior, referred to as "monopole oscillation", was very recently observed for the first time, in a gas of cold atoms.[2] Note that the collision integral does not appear in the equations (2), which means that they hold irrespective of whether there are inter-particle collisions or not. This is no longer true if the confining potential is not spherical symmetric—in which case the monopole oscillation couples to higher multipolar modes, which are damped by the collisions.[3]

[2] D. S. Lobser *et al.*, *Observation of a persistent non-equilibrium state in cold atoms*, Nature Phys. **11** (2015) 1009.

[3] The interested reader may have a look at D. Guéry-Odelin *et al.*, *Collective oscillations of a classical gas confined in harmonic traps*, Phys. Rev. A **60** (1999) 4851—note that there is a typo in Eq. (9).