Tutorial sheet 6

Discussion topic: You may start educating yourself on the relativistic Boltzmann equation by looking at part A of the book by De Groot *et al.*.

14. Free streaming

Show that in the absence of collision term and of external potential—which defines *free streaming*—, the solution of the (collisionless!) Boltzmann equation satisfies the equation

$$\bar{\mathsf{f}}(t,\vec{r},\vec{p}) = \bar{\mathsf{f}}(t\!=\!0,\vec{r}-\vec{v}t,\vec{p}) \tag{1}$$

with $\vec{v}$ the velocity corresponding to momentum $\vec{p}$. That is, the solution at time $t>0$ has a straightforward expression in terms of the initial condition at $t=0$.

15. Linearized Boltzmann equation

Consider the kinetic Boltzmann equation in absence of an external potential for neutral particles with mass m interacting elastically with each other. One easily checks that the Maxwell–Boltzmann distribution

$$\bar{\mathsf{f}}^{(0)}(\vec{p}) = n\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2} \mathrm{e}^{-\vec{p}^2/2mk_BT} \tag{2}$$

with constant T is a solution to the equation, whose integral over momentum[1] gives a uniform particle-number density n. In the following, we consider small perturbations

$$\bar{\mathsf{f}}(t,\vec{r},\vec{p}) = \bar{\mathsf{f}}^{(0)}(\vec{p})\big[1+h(t,\vec{r},\vec{p})\big] \tag{3}$$

away from the "equilibrium" solution (2), where quadratic terms in h will be systematically neglected.

i. Show that h obeys the linearized Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \vec{v}_1\cdot\vec{\nabla}_{\vec{r}}\right)h(t,\vec{r},\vec{p}_1) = \mathcal{I}_{\rm coll.}(h) \tag{4a}$$

where $\mathcal{I}_{\rm coll.}(h)$ is the (linear) collision operator

$$\mathcal{I}_{\rm coll.}(h) \equiv \int_{\vec{p}_2}\int_{\vec{p}_3}\int_{\vec{p}_4} \bar{\mathsf{f}}^{(0)}(\vec{p}_2)\Big[h(t,\vec{r},\vec{p}_3)+h(t,\vec{r},\vec{p}_4)-h(t,\vec{r},\vec{p}_1)-h(t,\vec{r},\vec{p}_2)\Big]\widetilde{w}(\vec{p}_1,\vec{p}_2\to\vec{p}_3,\vec{p}_4), \tag{4b}$$

where the same notations as in the lectures were used.

ii. Mathematical results

Consider spatially homogeneous perturbations $h(t,\vec{p})$. To investigate their behavior it is interesting to look at the eigenfunctions ψ_i and eigenvalues λ_i of the collision operator $\mathcal{I}_{\rm coll.}$, defined by

$$\mathcal{I}_{\rm coll.}(\psi_i) = \lambda_i\psi_i. \tag{5}$$

It will be assumed that the integral $\displaystyle\int_{\vec{p}_1}\bar{\mathsf{f}}^{(0)}(\vec{p}_1)\big[\psi_i(\vec{p}_1)\big]^2$ exists for every eigenfunction.

a) Show that $\lambda=0$ is a fivefold degenerate eigenvalue and give (the) corresponding eigenfunctions $\psi_1(\vec{p}_1)$, …, $\psi_5(\vec{p}_1)$ (disregarding any normalization).

Hint: You do not need to know the transition rate $\widetilde{w}$ to answer this question, which means that the eigenfunctions are determined by fundamental properties of the collisions.

[1]... with integration measure $\mathrm{d}^3\vec{p}/(2\pi\hbar)^3$

b) Show that all other eigenvalues are negative.

Hint: You may express λ_i in terms of the integral $\displaystyle\int_{\vec{p}_1} \bar{\mathsf{f}}^{(0)}(\vec{p}_1)\psi_i(\vec{p}_1)\mathcal{I}_{\rm coll.}(\psi_i)$, which you can transform as in the proof of the H-theorem.

c) Consider the linearized Boltzmann equation (4a) in the spatially homogeneous case. Assuming that the eigenfunctions form a complete set, write down the solution $h(t,\vec{p}_1)$ as a linear combination of the $\{\psi_i(\vec{p}_1)\}$. How can you interpret the quantities $-1/\lambda_i$ for the non-vanishing eigenvalues?

16. Boltzmann equation in cylindrical coordinates

Physical problems often have symmetries (axial, spherical...) which suggest the use of appropriate, non-Cartesian coordinates. For instance, one may wish to use cylindrical coordinates (r,θ,z) such that

$$\vec{r} = x\,\vec{\mathrm{e}}_x + y\,\vec{\mathrm{e}}_y + z\,\vec{\mathrm{e}}_z = r\,\vec{\mathrm{e}}_r + z\,\vec{\mathrm{e}}_z$$

with $r = \sqrt{x^2+y^2}$ and $\vec{\mathrm{e}}_r = \cos\theta\,\vec{\mathrm{e}}_x + \sin\theta\,\vec{\mathrm{e}}_y$. Let $\vec{\mathrm{e}}_\theta \equiv \vec{\mathrm{e}}_z \times \vec{\mathrm{e}}_r = -\sin\theta\,\vec{\mathrm{e}}_x + \cos\theta\,\vec{\mathrm{e}}_y$.

Show that the collisionless Boltzmann equation in absence of any external potential reads

$$\left[\frac{\partial}{\partial t} + v^r\frac{\partial}{\partial r} + \frac{v^\theta}{r}\frac{\partial}{\partial\theta} + v^z\frac{\partial}{\partial z} + \frac{(v^\theta)^2}{r}\frac{\partial}{\partial v^r} - \frac{v^r v^\theta}{r}\frac{\partial}{\partial v^\theta}\right]\bar{\mathsf{f}}_{\vec{v}}(t,r,\theta,z,v^r,v^\theta,v^z) = 0$$

where $\vec{v} = v^r\,\vec{\mathrm{e}}_r + v^\theta\,\vec{\mathrm{e}}_\theta + v^z\,\vec{\mathrm{e}}_z$, while $\bar{\mathsf{f}}_{\vec{v}}$ here denotes the density on position-velocity space, instead of phase space. Discuss the last two terms in the angular brackets.

Remark: The apparent singularity in $r=0$ of the factors $1/r$ in the above equation is obviously an artifact, since in fact $v^\theta = r\dot{\theta}$.