Tutorial sheet 3

Discussion topic: Reminder on equilibrium statistical mechanics: refresh your knowledge on the *phase space density* of a classical many-particle system.

6. Transport of pseudoscalar quantities

Energy or (total) particle number density are scalar quantities, so that the corresponding flux densities are polar vectors—i.e. they change sign under spatial parity inversion, as does the position vector $\vec{r}$. One may however also consider the transport of *pseudoscalar* quantities—as e.g. the difference between the numbers of particles with spin up and spin down along a given axis.

i. Let $\chi_{\rm ps}$ denote the density of a pseudoscalar quantity. Convince yourself that the conjugate intensive quantity $\mathcal{Y}_{\rm ps}$ is also pseudoscalar, while the associated flux density $\vec{\mathcal{J}}_{\rm ps}$ is an axial vector (or pseudovector).

ii. Consider a pseudoscalar quantity $\chi_{\rm ps}$ in a locally isotropic fluid, whose flow velocity $\vec{\mathsf{v}}$ is assumed to be a polar vector field. Can you write down a linear constitutive equation relating the flux density $\vec{\mathcal{J}}_{\rm ps}$ to the affinity $\vec{\nabla}\mathcal{Y}_{\rm ps}$? to the derivatives of the flow velocity? to the gradient of the (inverse of) temperature? Discuss the properties of the kinetic coefficients which you introduced.

7. Stress tensor in a fluid

Let π_{ij} denote the Cartesian components of the stress tensor in a fluid. Consider an infinitesimal cube of fluid of side $\mathrm{d}\ell$, where the sides are parallel to the axes of the (Cartesian) coordinate system.

i. Explain why the k-th component $\mathcal{M}_k$ of the torque exerted on the cube by the neighboring regions of the fluid obeys $\mathcal{M}_k \propto -\epsilon_{ijk}\pi_{ij}(\mathrm{d}\ell)^3$.

ii. Using dimensional considerations, write down the dependence of the moment of inertia I of the cube on $\mathrm{d}\ell$ and on the fluid mass density ρ.

iii. Using the results of the previous two questions, how does the rate of change of the angular velocity ω_k scale with $\mathrm{d}\ell$? How can you prevent this rate of change from diverging in the limit $\mathrm{d}\ell \to 0$?

8. Continuity equation for particle number

The single-particle phase-space density $f_1(t,\vec{r},\vec{p})$ of a collection of N particles is the ensemble average

$$f_1(t,\vec{r},\vec{p}) \equiv \left\langle \sum_{j=1}^{N} \delta^{(3)}\big(\vec{r}-\vec{r}_j(t)\big)\,\delta^{(3)}\big(\vec{p}-\vec{p}_j(t)\big) \right\rangle,$$

such that $f_1(t,\vec{r},\vec{p})\,\mathrm{d}^3\vec{r}\,\mathrm{d}^3\vec{p}$ is the number of particles in the phase space element $\mathrm{d}^3\vec{r}\,\mathrm{d}^3\vec{p}$ about the point $(\vec{r},\vec{p})$.
Show that the particle number density $n(t,\vec{r})$ and the particle current density $\vec{J}_N(t,\vec{r})$ are given by

$$n(t,\vec{r}) = \int f_1(t,\vec{r},\vec{p})\,\mathrm{d}^3\vec{p}, \qquad \vec{J}_N(t,\vec{r}) = \int f_1(t,\vec{r},\vec{p})\,\vec{v}\,\mathrm{d}^3\vec{p},$$

with $\vec{v}$ the velocity corresponding to momentum $\vec{p}$. Check that the continuity equation for particle number follows at once from these identities.