Tutorial sheet 9

Discussion topics: Markov processes; Fokker–Planck equation

19. Another view of the Fokker–Planck equation in one dimension

Consider an arbitrary one-dimensional Markovian process $X(t)$, taking its values in a real interval $[a,b]$, and such that the corresponding first two coefficients $\mathcal{M}_1(t,x)$, $\mathcal{M}_2(t,x)$ in the Kramers–Moyal expansion are actually independent of time.

i. Stationary solutions

Recall the form of the Fokker–Planck equation. Assuming that there is no flow of probability across the boundaries $x=a$ and $x=b$ ("reflecting boundary conditions"), write down the differential equation obeyed by the stationary solution $p^{\rm st.}_{X,1}(x)$ to the Fokker–Planck equation. Show that

$$p^{\rm st.}_{X,1}(x) = \frac{C}{\mathcal{M}_2(x)}\,\exp\!\left[2\int_a^x \frac{\mathcal{M}_1(x')}{\mathcal{M}_2(x')}\,{\rm d}x'\right], \tag{1}$$

where C is a constant which need not be computed. Why is this solution unique?

ii. Transforming the Fokker–Planck equation

Assume now that $\mathcal{M}_2$ is actually constant. Let $V(x) \equiv \dfrac{1}{2}[\mathcal{M}_1(x)]^2 + \dfrac{\mathcal{M}_2}{2}\dfrac{{\rm d}\mathcal{M}_1(x)}{{\rm d}x}$.

Perform the change of unknown function $p_{X,1}(t,x) = [p^{\rm st.}_{X,1}(x)]^{1/2}\psi(t,x)$ in the Fokker–Planck equation, where $p^{\rm st.}_{X,1}(x)$ is the stationary solution (1), and deduce the equation obeyed by $\psi(t,x)$. What do you recognize?

In the new language you just found, to which known problem is that of the Fokker–Planck equation for the Ornstein–Uhlenbeck process [$\mathcal{M}_1(x)=\gamma x$, $\mathcal{M}_2 = D$, $x\in\mathbb{R}$] equivalent?

20. Master equation for Markov processes

The purpose of this exercise is to derive a linear integrodifferential equation—which constitutes the differential form of the Chapman–Kolmogorov equation—for the transition probability and the single-time density of an (almost) arbitrary *homogeneous* Markov process $Y(t)$, i.e. a process for which the probability transition $p_{Y,1|1}(t_2,y_2\,|\,t_1,y_1)$ only depends on the time difference $\tau\equiv t_2-t_1$. In analogy with stationary processes, the latter will be denoted by $\mathcal{T}_{Y;\tau}(y_2\,|\,y_1)$.

We assume that for time differences τ much smaller than some time scale $\tau_{\rm c}$, the transition probability is of the form

$$\mathcal{T}_{Y;\tau}(y_2\,|\,y_1) = [1-\gamma(y_1)\,\tau]\,\delta(y_2-y_1) + \Gamma(y_2\,|\,y_1)\,\tau + o(\tau), \tag{2a}$$

where $o(\tau)$ denotes a term which is much smaller than τ in the limit $\tau\to 0$. The nonnegative quantity $\Gamma(y_2\,|\,y_1)$ is the transition rate from y_1 to y_2, and γ is its integral over y_2

$$\gamma(y_1) = \int \Gamma(y_2\,|\,y_1)\,{\rm d}y_2. \tag{2b}$$

i. Compute the integral of the transition probability $\mathcal{T}_{Y;\tau}(y_2\,|\,y_1)$ over final states y_2.

ii. Master equation

Starting from the Chapman–Kolmogorov equation

$$\mathcal{T}_{Y;\tau+\tau'}(y_3\,|\,y_1) = \int \mathcal{T}_{Y;\tau'}(y_3\,|\,y_2)\,\mathcal{T}_{Y;\tau}(y_2\,|\,y_1)\,{\rm d}y_2,$$

and assuming that $\tau'\ll\tau_{\rm c}$—note that no assumption on τ is needed—, show that after leaving aside a negligible term, one obtains

$$\mathcal{T}_{Y;\tau+\tau'}(y_3\,|\,y_1) = \big[1-\gamma(y_3)\,\tau'\big]\,\mathcal{T}_{Y;\tau}(y_3\,|\,y_1) + \tau'\!\int \Gamma(y_3\,|\,y_2)\,\mathcal{T}_{Y;\tau}(y_2\,|\,y_1)\,\mathrm{d}y_2,$$

Check that this leads in the limit $\tau' \to 0$ to the integrodifferential equation

$$\frac{\partial\mathcal{T}_{Y;\tau}(y_3\,|\,y_1)}{\partial\tau} = -\gamma(y_3)\mathcal{T}_{Y;\tau}(y_3\,|\,y_1) + \int \Gamma(y_3\,|\,y_2)\,\mathcal{T}_{Y;\tau}(y_2\,|\,y_1)\,\mathrm{d}y_2,$$

and eventually, after invoking Eq. (2b) and relabelling the variables, to the *master equation*

$$\frac{\partial\mathcal{T}_{Y;\tau}(y\,|\,y_0)}{\partial\tau} = \int \big[\Gamma(y\,|\,y')\,\mathcal{T}_{Y;\tau}(y'\,|\,y_0) - \Gamma(y'\,|\,y)\,\mathcal{T}_{Y;\tau}(y\,|\,y_0)\big]\,\mathrm{d}y'. \tag{3}$$

Note that this evolution equation has the structure of a balance equation, with a gain term, involving the rate $\Gamma(y\,|\,y')$, and a loss term depending on the rate $\Gamma(y'\,|\,y)$.

iii. Evolution equation for the single-time probability density

Starting from the consistency condition

$$p_{Y,1}(\tau, y) = \int \mathcal{T}_{Y;\tau}(y\,|\,y_0)\,p_{Y,1}(t\!=\!0, y_0)\,\mathrm{d}y_0, \tag{4}$$

show that the above master equation leads to

$$\frac{\partial p_{Y,1}(\tau, y)}{\partial\tau} = \int \big[\Gamma(y\,|\,y')\,\mathcal{T}_{Y;\tau}(y'\,|\,y_0) - \Gamma(y'\,|\,y)\,\mathcal{T}_{Y;\tau}(y\,|\,y_0)\big]\,p_{Y,1}(t\!=\!0, y_0)\,\mathrm{d}y_0\,\mathrm{d}y'.$$

Check that this then leads to the evolution equation

$$\frac{\partial p_{Y,1}(\tau, y)}{\partial\tau} = \int \big[\Gamma(y\,|\,y')\,p_{Y,1}(\tau, y') - \Gamma(y'\,|\,y)\,p_{Y,1}(\tau, y)\big]\,\mathrm{d}y', \tag{5}$$

which is formally identical to the master equation for $\mathcal{T}_{Y;\tau}$.