Tutorial sheet 8

Discussion topic: Langevin model of Brownian motion

16. Stochastic processes

Let $Y(t)$ be a stochastic process and $p_{Y,n}$ resp. $p_{Y,n|m}$ its n-point resp. conditional n-point densities.

i. Starting from the expression of Bayes' theorem, show that for all integers $n \geq 2$ one can write

$$p_{Y,n}(t_1, y_1; \ldots; t_n, y_n) = p_{Y,1|n-1}(t_n, y_n \,|\, t_1, y_1; \ldots; t_{n-1}, y_{n-1}) \\ \times p_{Y,1|n-2}(t_{n-1}, y_{n-1} \,|\, t_1, y_1; \ldots; t_{n-2}, y_{n-2}) \cdots p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)\, p_{Y,1}(t_1, y_1),$$

where the n instants $t_1, t_2, \ldots, t_n$ are all different, but otherwise arbitrary.

ii. Consider the previous identity for $n = 3$. Using the consistency condition which expresses $p_{Y,m}$ as an integral of $p_{Y,n}$ with $n > m$, here with $m = 2$, show that the single-time conditional density $p_{Y,1|1}$ obeys the integral-functional equation

$$p_{Y,1|1}(t_3, y_3 \,|\, t_1, y_1) = \int p_{Y,1|2}(t_3, y_3 \,|\, t_1, y_1; t_2, y_2)\, p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)\, \mathrm{d}y_2. \tag{1}$$

Generalize this relation to an equation for $p_{Y,1|n}$ involving $p_{Y,1|n+1}$.

17. Examples of Markov processes

In the lecture, we shall introduce a specific, important class of stochastic processes, the so-called *Markov processes*. These are entirely determined by their single-time density $p_{Y,1}$ and their conditional probability density $p_{Y,1|1}$, which is referred to as *transition probability* and obeys the *Chapman–Kolmogorov equation*

$$p_{Y,1|1}(t_3, y_3 \,|\, t_1, y_1) = \int p_{Y,1|1}(t_3, y_3 \,|\, t_2, y_2)\, p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1)\, \mathrm{d}y_2 \quad \text{for } t_1 < t_2 < t_3, \tag{2}$$

to be compared with Eq. (1) in exercise **16.ii**.

i. Wiener process

The stochastic process defined by the "initial condition" $p_{Y,1}(t=0, y) = \delta(y)$ for $y \in \mathbb{R}$ and the transition probability $(0 < t_1 < t_2)$

$$p_{Y,1|1}(t_2, y_2 \,|\, t_1, y_1) = \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp\left[-\frac{(y_2 - y_1)^2}{2(t_2 - t_1)}\right]$$

is called *Wiener process*.

Check that this transition probability obeys the Chapman–Kolmogorov equation, and that the probability density at time $t > 0$ is given by

$$p_{Y,1}(t, y) = \frac{1}{\sqrt{2\pi t}}\, \mathrm{e}^{-y^2/2t}.$$

Remark: Note that the above single-time probability density is solution of the diffusion equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial y^2}$$

with diffusion coefficient $D = \frac{1}{2}$.

ii. Ornstein–Uhlenbeck process

The so-called *Ornstein–Uhlenbeck process* is defined by the time-independent single-time probability density

$$p_{Y,1}(y) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-y^2/2}$$

and the transition probability ($\tau > 0$)

$$p_{Y,1|1}(t+\tau, y\,|\,t, y_0) = \frac{1}{\sqrt{2\pi(1-\mathrm{e}^{-2\tau})}}\,\exp\left[-\frac{(y-y_0\mathrm{e}^{-\tau})^2}{2(1-\mathrm{e}^{-2\tau})}\right].$$

a) Check that this transition probability fulfills the Chapman–Kolmogorov equation, so that the Ornstein–Uhlenbeck process is Markovian. Show that the process is also Gaussian, stationary, and that its autocorrelation function is $\kappa(\tau) = \mathrm{e}^{-\tau}$.

b) What is the large-τ limit of the transition probability? And its limit when τ goes to 0^+?

c) Viewing the above transition probability as a function of τ and y, can you find a partial differential equation, of which it is a (fundamental) solution?

Hint: Let yourself be inspired(?) by the remark at the end of question **i.**

18. Characteristic functional of a stochastic process

In the lecture, the characteristic functional associated with a stochastic process $Y_X(t)$ has been defined as

$$G_Y[k(t)] \equiv \left\langle \exp\left[\mathrm{i}\int k(t)\,Y(t)\,\mathrm{d}t\right]\right\rangle,$$

with $k(t)$ a test function.

Expand the exponential in power series of k and express the characteristic functional in terms of the n-time moments. How would you write the moment $\langle Y(t_1)Y(t_2)\cdots Y(t_n)\rangle$ as function of $G_Y[k(t)]$?